

*Centre of Mass and It's Motion:

The centre of mass of a system of particles or body is that point at which the total mass of a body can be assumed to be concentrated. For the regular geometrical shape of the bodies, the centre of mass is that point about which the body is symmetrical. For the objects of irregular shape, it depends on the distribution of mass.

Let us consider a system of N particles, having masses $m_1, m_2, \dots, m_N$, and instantaneous position vectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$ respectively in a frame of reference XYZ (fig. 3.1). The centre of mass of this system at this instant is the point whose position vector $\vec{R}$ is given by

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_N\vec{r}_N, \quad \dots (1)$$

where M is the total mass of the system, i.e.,

$$M = \sum m_i = m_1 + m_2 + \dots + m_N.$$

Therefore,
$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_N\vec{r}_N}{m_1 + m_2 + \dots + m_N} = \frac{\sum_{i=1}^N m_i\vec{r}_i}{\sum_{i=1}^N m_i}. \quad \dots (2)$$

The sum $\sum m_i\vec{r}_i$ is called the **first moment of mass** for the system.

If the coordinates of centre of mass of the system are (X, Y, Z) , then

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}. \quad \dots (3)$$

But $\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ and then comparing (2) and (3), we get

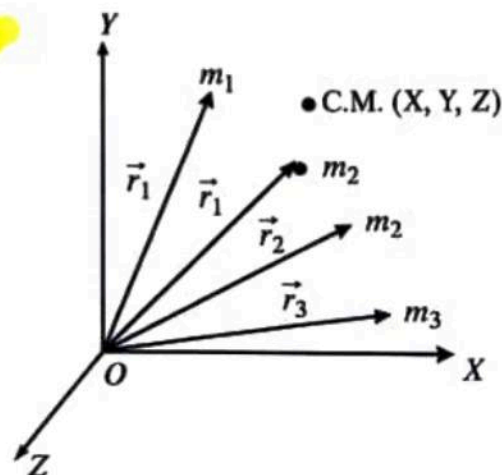


Fig. 3.1. Centre of mass of a system of particles.

$$X = \frac{m_1 x_1 + m_2 x_2 + \dots + m_N x_N}{m_1 + m_2 + \dots + m_N} = \frac{\sum m_i x_i}{\sum m_i}; \quad Y = \frac{m_1 y_1 + m_2 y_2 + \dots + m_N y_N}{m_1 + m_2 + \dots + m_N} = \frac{\sum m_i y_i}{\sum m_i};$$

$$Z = \frac{m_1 z_1 + m_2 z_2 + \dots + m_N z_N}{m_1 + m_2 + \dots + m_N} = \frac{\sum m_i z_i}{\sum m_i}, \quad \dots (4)$$

Special case : If the centre of mass is situated at the origin of the frame of reference, then $\vec{R} = 0$ and from eq. (2), we have

$$\sum m_i \vec{r}_i = 0 \quad \dots (5)$$

Hence, the sum of moments of all the particles about the centre of mass is equal to zero.

Motion of centre of mass : From eq. (1),

$$M \vec{R} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_N \vec{r}_N.$$

Differentiating eq. (1) with respect to time t , we get

$$M \frac{d\vec{R}}{dt} = m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + \dots + m_N \frac{d\vec{r}_N}{dt}.$$

Now, writing $\vec{v}$ for the velocity $d\vec{R}/dt$ of the centre of mass and $\vec{v}_1, \vec{v}_2, \dots$, for the velocities of the particles, we have

$$M \vec{v} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_N \vec{v}_N \quad \dots (6)$$

or
$$M \vec{v} = \sum m_i \vec{v}_i = \vec{P}.$$

or
$$\vec{v} = \frac{\sum m_i \vec{v}_i}{M} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_N \vec{v}_N}{m_1 + m_2 + \dots + m_N}. \quad \dots (7)$$

This equation gives the **velocity of the centre of mass**.

Middle term of eq. (6) is the total linear momentum of the system ($\vec{P}$). Thus the total linear momentum of a system is equal to the product of the total mass and velocity of the centre of mass. In absence of external forces, the total linear momentum ($\sum m_i \vec{v}_i$) remains constant, hence from eq. (6), we have

$$M \vec{v} = \text{a constant} \quad \text{or} \quad \vec{v} = \text{a constant}. \quad \dots (8)$$

This is a remarkable property of the centre of mass that its velocity remains constant in the absence of external forces. F

Ex. 1. The mass and position vectors of two particles are respectively 6 kg, $(6\vec{i} + 7\vec{j})$ and 2 kg, $(2\vec{i} + 10\vec{j} - 8\vec{k})$ m. Find the position vector of centre of mass.

Sol. Position vector of the centre of mass of two-particle system is given by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{6(6\vec{i} + 7\vec{j}) + 2(2\vec{i} + 10\vec{j} - 8\vec{k})}{6 + 2} = \frac{40\vec{i} + 62\vec{j} - 16\vec{k}}{8} = 5\vec{i} + 7.75\vec{j} - 2\vec{k} \text{ m.}$$

Ex. 2. If the centre of mass of three particles of masses 10, 20 and 30 gm be at the point (1, 1, 1) m, then where should a fourth particle of 40 gm be placed so that the combined centre of mass be at (0, 0, 0) ? (Agra 2010)

Sol.
$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \quad \text{or} \quad 1 = \frac{0.01x_1 + 0.02x_2 + 0.03x_3}{0.01 + 0.02 + 0.03}$$

or
$$x_1 + 2x_2 + 3x_3 = 6 \quad \dots(i)$$

Suppose 40 gm mass is placed at a point (x_4, y_4, z_4) so that the combined C.M. is at (0, 0, 0). Now,

$$0 = \frac{0.01x_1 + 0.02x_2 + 0.03x_3 + 0.04x_4}{0.01 + 0.02 + 0.03 + 0.04}$$

or
$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \quad \dots(ii)$$

Subtracting eq. (i) from eq. (ii), we have

$$4x_4 = -6 \quad \text{or} \quad x_4 = -\frac{3}{2}$$

In the same way, $y_4 = -\frac{3}{2}$ and $z_4 = -\frac{3}{2}$.

Therefore 40 gm mass should be placed at $(-3/2, -3/2, -3/2)$ m.

Ex. 3. Two particles of masses 100 gm and 300 gm have at a given time position $(2\vec{i} + 5\vec{j} + 13\vec{k})$ m and $(-6\vec{i} + 4\vec{j} - 2\vec{k})$ m respectively, and velocities $(10\vec{i} - 7\vec{j} - 3\vec{k})$ and $(7\vec{i} - 9\vec{j} + 6\vec{k})$ m/sec respectively. Deduce (i) the instantaneous position of the centre of mass, (ii) the velocity of the second particle in a frame of reference travelling with the centre of mass. (Kanpur 2005)

Sol. (i)
$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{0.1(10\vec{i} - 7\vec{j} - 13\vec{k}) + 0.3(-6\vec{i} + 4\vec{j} + 2\vec{k})}{0.1 + 0.3} = \frac{-16\vec{i} + 17\vec{j} + 7\vec{k}}{4} \text{ m.}$$

(ii)
$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{0.1(10\vec{i} - 7\vec{j} - 3\vec{k}) + 0.3(7\vec{i} - 9\vec{j} + 6\vec{k})}{0.1 + 0.3} = \frac{31\vec{i} - 34\vec{j} + 15\vec{k}}{4} \text{ m/s.}$$

Velocity of the second particle in the centre of mass frame of reference

$$= \vec{v}_2 - \vec{V} = 7\vec{i} - 9\vec{j} + 6\vec{k} - \frac{31\vec{i} - 34\vec{j} + 15\vec{k}}{4} = -\frac{3\vec{i} + 2\vec{j} + 9\vec{k}}{4} \text{ m/sec.}$$

Conservation of Linear Momentum

If a force $\vec{F}$ is acting on a particle of mass m , then according to Newton's second law of motion, we have

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) \quad \dots(1)$$

where $\vec{p} = m\vec{v}$ is the linear momentum of the particle. Momentum is a vector quantity.

If the external force acting on the particle is zero, then

$$\vec{F} = \frac{d\vec{p}}{dt} = 0 \quad \text{or} \quad \vec{p} = m\vec{v} = \text{a constant.} \quad \dots(2)$$

Thus, in absence of an external force, the linear momentum of a particle remains constant. This is principle of conservation of linear momentum for a single particle.

* Conservation of linear momentum for a system of two particles : Suppose that in an isolated system there are two particles of masses m_1 and m_2 . No external force is acting on the system. However both particles are exerting interaction forces on each other, as in the collision of two particles. We assume that these forces satisfy Newton's third law of motion and we say that these forces are *Newtonian type*. According to Newton's third law of motion, the force $\vec{F}_{12}$ which first particle (1) exerts upon the other (2) is the negative of the force $\vec{F}_{21}$ exerted by second (2) on first (1); i.e.,

$$\vec{F}_{12} = -\vec{F}_{21} \quad \text{or} \quad \vec{F}_{12} + \vec{F}_{21} = 0 \quad \dots(3)$$

If at any instant t , the velocities of the particles are $\vec{v}_1$ and $\vec{v}_2$, then their momenta will be $\vec{p}_1 = m_1\vec{v}_1$ and $\vec{p}_2 = m_2\vec{v}_2$ respectively. Now, according to Newton's second law of motion,

$$\vec{F}_{12} = \frac{d\vec{p}_1}{dt} = \frac{d}{dt}(m_1\vec{v}_1), \text{ and} \quad \vec{F}_{21} = \frac{d\vec{p}_2}{dt} = \frac{d}{dt}(m_2\vec{v}_2)$$

Hence eq. (3) assumes the form

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$$

$$\text{i.e.,} \quad \vec{p}_1 + \vec{p}_2 = \text{a constant} \quad \text{or} \quad m_1\vec{v}_1 + m_2\vec{v}_2 = \text{a constant} \quad \dots(4)$$

Thus in absence of external forces on a system of two particles, their total linear momentum is conserved. This is the law of conservation of linear momentum for a system of two particles. Hence we can say that the conservation of linear momentum is equivalent to Newton's third law.

Central Force

When a force acts on a particle in such a way that it is always directed towards or away from a fixed point and its magnitude depends only upon the distance (r) from the point, then such a force is called a **central force**. Thus, a central force is represented by

$$\vec{F} = F(r) \hat{r} = F(r) \frac{\vec{r}}{r} \quad \dots(1)$$

where $F(r)$ is any function of distance r and $\hat{r} = \frac{\vec{r}}{r}$ is a unit vector along $\vec{r}$ from the fixed centre.

** Example of Central Forces:

(1) Gravitational force of attraction between two masses :

According to Newton's law of universal gravitation, the force of attraction between any two masses m_1 and m_2 , separated by a distance r , is given by (fig. 5.3) :

$$F = -\frac{Gm_1m_2}{r^2} \hat{r}$$

where $\hat{r}$ is the position vector from m_1 to m_2 , r is the distance between the two masses and $G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$ is the gravitational constant.

(2) Electrostatic force between two charges :

According to Coulomb's law, the electrostatic force between two charges q_1 and q_2 , separated by a distance r , is given by

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1q_2}{r^2} \hat{r}$$

where $\hat{r}$ is the unit vector from q_1 to q_2 and ϵ_0 is the permittivity of free space.

(3) Elastic force in spring-mass system :

Consider a spring whose one end is fixed to a rigid support and the other end is attached to a block of mass m . If the spring is stretched or compressed by a distance x , the elastic force is given by

$$F = -kx \hat{x}$$

where k is the force constant, $\hat{x}$ is the unit vector along the spring and x is the displacement.

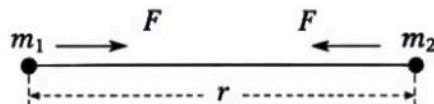


Fig. 7.3. Gravitational force of attraction between two masses.

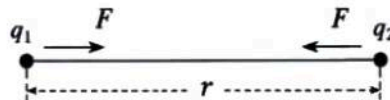


Fig. 7.4. Electrostatic force between two charges.

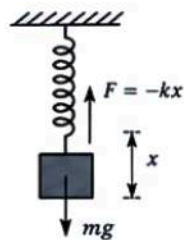


Fig. 7.5. Elastic force in spring-mass system.

*Reduced Mass:

Reduced mass is an effective inertial mass appearing in the two-body problem of Newtonian mechanics. When two bodies, with masses m_1 and m_2 , interact with each other (for example, through gravity or an electrostatic force), they both move around their common centre of mass. By using the concept of reduced mass, this two-body problem can be mathematically transformed into a simpler one-body problem, where a single particle of reduced mass μ moves around a stationary centre.

The reduced mass, typically denoted by the Greek letter μ (mu), is defined by the following equation:

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

- In **hydrogen atom**, an electron of mass m revolves round a proton of much heavier mass M . The reduced mass of the hydrogen atom will be given by

$$\mu = \frac{mM}{m+M} = m \left(1 + \frac{m}{M} \right)^{-1} = m \left(1 - \frac{m}{M} \right).$$

But $\frac{\text{mass of the electron}}{\text{mass of the proton}} = \frac{m}{M} = \frac{1}{1836}.$

$$\mu = m \left(1 - \frac{1}{1836} \right).$$

Hence in case of hydrogen atom, if the energy, period etc. are calculated by assuming that an electron moves round a fixed proton, there will occur some error. To get the correct results, m should be replaced by the reduced mass μ in the expressions of energy, period etc. Actually, in case of hydrogen atom, this error will be very small, because

$$m \left(1 - \frac{1}{1836} \right) \cong m \text{ nearly}$$

- Positronium** is a temporary combination of a positron and an electron similar to a hydrogen atom. A positron is a particle which has mass equal to the electron mass but has positive charge. The reduced mass of the positronium is given by

$$\boxed{\mu' = \frac{mn}{m+n} = \frac{m}{2}}. \text{ Since, } (m = \text{mass of the electron} = \text{mass of positron})$$

Thus the reduced mass of the positronium particle is one-half the mass of the electron.

Example 1. Calculate the reduced mass of hydrogen molecule (H_2). The mass of hydrogen atom is 1.7×10^{-27} kg.

Sol. A molecule of hydrogen consists of two hydrogen atoms. For a hydrogen molecule, $m_1 = m_2 = m$,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m}{2}$$

Here, $m = 1.7 \times 10^{-27}$ kg

$$\therefore \mu = \frac{1.7 \times 10^{-27}}{2} = 8.5 \times 10^{-28} \text{ kg.}$$

****Two Body Problem Reduced to One Body Problem:**

Let us consider two particles of masses m_1 and m_2 , whose instantaneous position vectors in an inertial frame with origin O , are $\vec{r}_1$ and $\vec{r}_2$ [fig. 7.6]. Hence the vector distance of m_1 from m_2 is $\vec{r} = \vec{r}_1 - \vec{r}_2$. If m_1 and m_2 interact via central force collinear with vector $\vec{r}$, then the forces acting on the two particles are given by

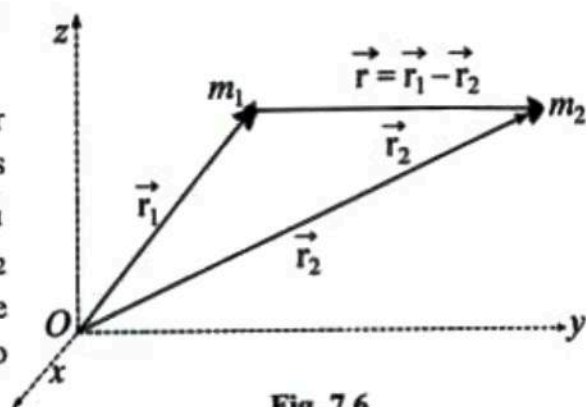


Fig. 7.6.

$$\vec{F}_{12} = m_1 \frac{d^2 \vec{r}_1}{dt^2} \quad \text{and} \quad \vec{F}_{21} = m_2 \frac{d^2 \vec{r}_2}{dt^2} \quad \dots(1)$$

But according to Newton's third law of motion

$$\vec{F}_{12} = -\vec{F}_{21} = \vec{F} \text{ (say)}, \quad \therefore \frac{d^2 \vec{r}_1}{dt^2} = \frac{\vec{F}}{m_1} \quad \text{and} \quad \frac{d^2 \vec{r}_2}{dt^2} = -\frac{\vec{F}}{m_2} \quad \dots(2)$$

Hence
$$\frac{d^2 \vec{r}_2}{dt^2} (\vec{r}_1 - \vec{r}_2) = \vec{F} \left(\frac{1}{m_1} + \frac{1}{m_2} \right).$$

If $\hat{r}$ is the unit vector along $\vec{r} = \vec{r}_1 - \vec{r}_2$ then $\vec{F} = F\hat{r}$, where F is negative for attractive force and positive for repulsive force. For central force, F is any function of distance r . Now,

$$\frac{d^2 \vec{r}}{dt^2} = \left(\frac{1}{m_1} + \frac{1}{m_2} \right) F\hat{r} \quad \dots(2)$$

If we put $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ in this equation, then

$$\frac{d^2 \vec{r}}{dt^2} = \frac{\vec{F}}{\mu} = \frac{F\hat{r}}{\mu} \quad \dots(3)$$

or

$$\vec{F} = \mu \frac{d^2 \vec{r}}{dt^2} \quad \dots(4)$$

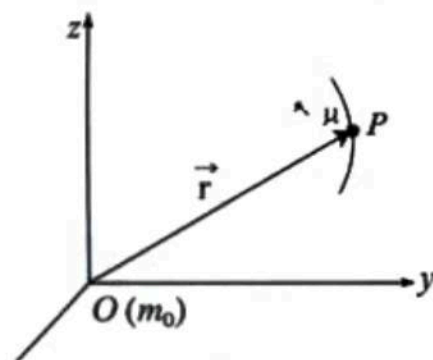


Fig. 7.7.

This equation represents a one-body problem, because it is similar to the equation of motion of a single particle of mass μ at vector $\vec{r}$ from one of the two particles, considered as centre of force O^* (fig. 7.7). Thus, the original two-body problem, involving the calculation of $\vec{r}_1$ and $\vec{r}_2$ has been reduced to the calculation of a single vector $\vec{r}$ as function of time. The quantity $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is therefore called the reduced mass of the two particles, having masses m_1 and m_2 . The reduced mass μ has a value less than m_1 or m_2 .

*Kepler's laws of planetary motion:

Kepler's laws of planetary motion are three scientific rules that describe how planets move around the Sun.

First Law: Law of Elliptical orbits-

Every Planet moves in an elliptical orbit around the sun, the sun being at one of the two foci (focal points).

According to this law, the path of a planet around sun is in the form of an ellipse and the sun is at one of the foci. Different planets of the sun move in different orbits about it.

Second Law: Law of Equal Areas-

The radius vector, drawn from a planet to the Sun, sweeps out equal areas in equal times, i.e, the areal velocity of the radius vector is constant.

Planets move faster when they are close to the Sun and slower when they are far away. They cover different distances along the path, but the "pie slice" area of space they cross is always the same size for the same time frame.

Third Law: Law of Harmonies-

The square of the period of revolution of the planet around the sun(T^2) is directly proportional to the cube of the semi-major axis (a^3 , or average distance from the Sun) of the ellipse. Thus,

$$T^2 \propto a^3$$

*Areal Velocity of a Planet around the Sun is Constant:

Motion under central force – Areal velocity : A central force, acting on a particle, depends only upon the magnitude of its distance from a fixed centre. If $\vec{r}$ is the instantaneous position vector of the particle relative to the fixed centre, then the central force is represented by the relation,

$$\vec{F} = f(r)\hat{r}, \text{ where } f(r) \text{ is a scalar function of distance } r \text{ and } \hat{r} = \vec{r}/r.$$

The torque acting on the particle is

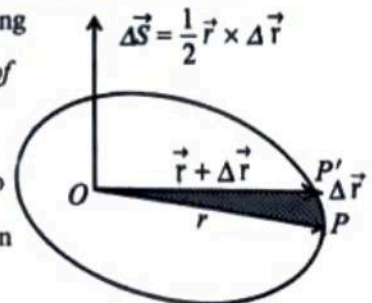
$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = f(r)\left[\frac{\vec{r}}{r} \times \vec{r}\right] = 0 \quad [\because \vec{r} \times \vec{r} = 0]$$

So that
$$\vec{L} = \vec{r} \times m\vec{v} = \text{constant}$$

• Thus, the angular momentum of a particle, moving under central force, is conserved. Moreover, if $\vec{L}$ is constant, it should always be perpendicular to the plane containing $\vec{r}$ and $\vec{v}$. This means that the path of the particle in the influence of a central force lies in a plane.

Now, let O be the centre of force, when the vector $\vec{r}$ changes to $\vec{r} + \Delta\vec{r}$, the vector area $\Delta\vec{S}$ swept by the radius vector in this time is

$$\Delta\vec{S} = \frac{1}{2} \vec{r} \times \Delta\vec{r} \quad \dots(2)$$



This area is swept out in Δt time, therefore dividing both the sides of this equation by Δt and taking limit $\Delta t \rightarrow 0$, then

$$\frac{d\vec{s}}{dt} = \frac{1}{2} \vec{r} \times \vec{v} = \frac{\vec{L}}{2m} \quad \dots\dots\dots(3)$$

This expression gives the areal velocity $\frac{d\vec{s}}{dt}$ of the particle. But $\vec{L}$ is constant for central forces, so from equation (3) the areal velocity remains constant, when the particle moves in the influence of a central force.

*Geostationary Satellite:

Geostationary satellite : If a satellite with period of revolution, $T = 24$ hours is put in a circular orbit with its plane coinciding with equatorial plane, moving from west to east, then to an observer on the earth the satellite will be seen stationary. Such a satellite is called *geostationary satellite*.

Let h be the height (altitude) of a geostationary satellite from the surface of the earth. Then, the distance of the satellite from the centre of the earth is $r = R + h$ (fig. 7.17). Hence, the period of revolution of such a satellite is given by [from equation (4)].

$$T = \frac{2\pi r}{R} \sqrt{\frac{r^3}{g}} = 2\pi \sqrt{\frac{r^3}{gR^2}}.$$

But $T = 24$ hours $= 24 \times 60 \times 60$ sec and $R = 6.4 \times 10^6$ m and $g = 9.8$ m/sec. Hence,

$$2 \times 3.14 \times \left[\frac{r^3}{9.8 \times (6.4 \times 10^6)^2} \right]^{1/2} = 24 \times 60 \times 60$$

whence $r = 4.2 \times 10^7$ m.

$\therefore$ Height (altitude) of the geostationary satellite

$$h = r - R = 4.2 \times 10^7 - 6.4 \times 10^6 = 3.56 \times 10^7 \text{ m} = \underline{3.56 \times 10^4 \text{ km.}}$$

Uses of Geostationary Satellite :

- (1) In the long range transmission of radio and television programmes.
- (2) In sending and receiving signals in space flights.
- (3) In the study of earth's upper atmosphere.
- (4) In the study of the cosmic radiations coming from the sun and outside.
- (5) In the study of the weather.

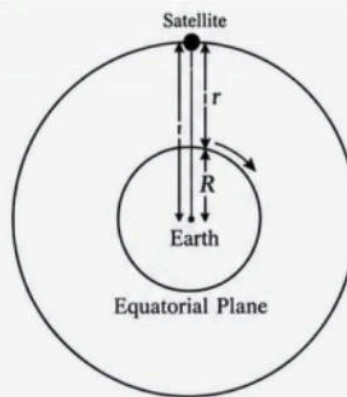


Fig. 7.17.

*Global Positioning System:

GPS stands for Global Positioning System. It is a satellite-based radio navigation system used on land, sea, and air to determine the exact location, time and velocity irrespective of weather conditions. The US military first used it in the year 1960.

Components of a GPS system-

GPS is a system and it is made up of three parts: *satellites*, *ground stations*, and *receivers*.

Following are the functionalities of each of these parts:

*Satellites act like the stars in constellations, and we know where they are because they invariably send out signals.

*The ground stations make use of the radar to make sure the satellites are actually where we think they are.

*A receiver is a device that we might find in our phone or in our car and it constantly seeks for the signals from the satellites. The receiver figures out how far away they are from some of them. Once the receiver calculates its distance from four or more satellites, it knows exactly where we are.

(एक रिसीवर एक ऐसा उपकरण है जो हमारे फोन या कार में पाया जा सकता है, और यह लगातार उपग्रहों (satellites) से आने वाले संकेतों (सिग्नल्स) की तलाश करता है। रिसीवर यह पता लगाता है कि वे उपग्रह उससे कितनी दूर हैं। एक बार जब रिसीवर चार या उससे अधिक उपग्रहों से अपनी दूरी की गणना कर लेता है, तो उसे ठीक-ठीक पता चल जाता है कि हम इस समय कहाँ हैं।)

Dynamics of a Rigid Body

Definition of Rigid Body

A rigid body is defined as a system of particles in which the distances between all pairs of particles remain fixed throughout the motion. Thus, in case of a rigid body if r_{ij} is the distance between any two and j th particles then

$$r_{ij} = \text{constant}$$

In a solid the distances between its constituent particles change to some extent when there is strain due to external forces. If such changes are negligibly small, the solid may be considered as a rigid body. Thus, a rigid body will not undergo any change of shape or size due to the application of external forces.

Moment of Inertia about an Axis

The *moment of inertia* of a mass point about an axis is defined as the product of its mass and the square of its distance from the axis of rotation. If a particle of mass m be situated at a distance r from the given axis, its moment of inertia I , therefore, is given by

$$I = mr^2 \quad \dots(1)$$

Consider a rigid body of mass m capable of rotating about an axis AB [Fig.]. Assume that the body is composed of a very large number of small particles of masses $m_1, m_2, m_3, \dots$ etc. at distances $r_1, r_2, r_3, \dots$ etc. respectively from the axis AB . Then the moment of inertia of the particle of mass m_1 , about the axis AB is $m_1 r_1^2$, that of the particle of the mass m_2 is $m_2 r_2^2$ and so on. Hence the moment of inertia I of the whole body about the given axis is equal to the sum of the quantities $m_1 r_1^2, m_2 r_2^2, m_3 r_3^2, \dots$ etc. i.e.,

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots$$
$$I = \sum mr^2 \quad \dots(2)$$

Hence, the *moment of inertia* of a body about an axis is defined as the sum of the products of the masses of different particles, supposed to be constituting the body, and the square of their respective distances from the axis of rotation. In C.G.S. system, the unit of moment of inertia is **gm-cm²** and in S.I. system it is **kg-m²** and the dimensional formula is **[ML²]**.

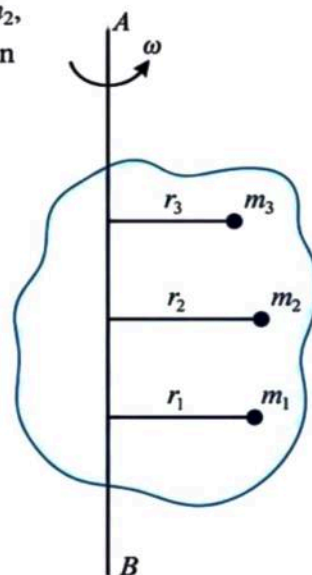


Fig. . Moment of inertia about an axis.

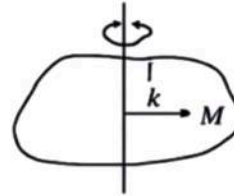
Radius of gyration

Whatever may be the shape of a body, it is always possible to find a distance from the axis of rotation at which whole mass of the body could be assumed to be concentrated and even then its moment of inertia about that axis remains unaltered. For this, if the whole mass M of a body be supposed to be concentrated at a distance k from the axis of rotation, then clearly

$$I = Mk^2 = \sum mr^2$$

or

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{\sum mr^2}{M}}$$



This quantity k is called **radius of gyration** of the body about the axis of rotation. Thus, *the radius of gyration of a body, rotating about a given axis, is equal to the distance, from the axis of rotation, the square of which, when multiplied by the total mass of the body, will give the moment of inertia of the body about that axis.*

*Physical Significance of Moment of Inertia:

In case of rotatory motion, the quantities force and mass of translatory motion are replaced by torque and moment of inertia respectively. This is the physical significance of moment of inertia.

Kinetic Energy of a Rotating Body about an Axis

The kinetic energy of a rotating body about an axis depends not only upon its mass and angular velocity but also depends upon the position of the axis and the distribution of mass about that axis.

Suppose a body of mass M is rotating about an axis AB . Assume that the body is composed of a very large number of small particles [Fig. 5.6]. Consider a particle P of elementary mass m_i at a distance r_i from the axis. If ω is the angular velocity, the linear velocity of the particle P will be $r_i\omega$.

Therefore, the kinetic energy of rotation of the particle P is $\frac{1}{2} m_i r_i^2 \omega^2$. Hence the kinetic energy (K.E.) of rotation of the whole body is given by

$$K = \sum \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} \omega^2 \sum m_i r_i^2,$$

since the angular velocity ω is the same for all the constituent particles of the body.

Now, $\sum m_i r_i^2 = I$, the moment of inertia about the axis AB ,

$$\therefore \text{K.E. of rotation } K = \frac{1}{2} I \omega^2. \quad \dots(1)$$

If $\omega = 1$, obviously $I = 2 \times K$.

Hence, the moment of inertia of a body may also be defined as twice its kinetic energy of rotation, the angular velocity of the body being unity.

But angular momentum, $L = I\omega$, ... (2)

where L is along the axis of rotation,

$$L = \sqrt{2KI} = \sqrt{2KL} \quad \text{or} \quad K = \frac{L^2}{2I}. \quad \dots(3)$$

This is the relation between the rotational kinetic energy and angular momentum about the same axis.

Rolling bodies : Let us calculate the kinetic energy of a rolling body. When a body with circular symmetry (e.g., cylinder, wheel, disc, sphere etc.) rolls on a plane surface, its motion is a combination of translation and rotation. At any instant the axis normal to the plane of the diagram through the point of contact P is the axis of rotation (Fig. 5.7). If the speed of the centre of mass is V (relative to an observer fixed on the surface), the instantaneous angular speed about

an axis through P is $\omega = V/R$, where R is the radius of the body. Therefore, at that instant all the particles of the rigid body are moving with the same angular speed ω about the axis through P and the motion of the body is equivalent to a pure rotation. Thus, the total kinetic energy is

$$K = \frac{1}{2} I_P \omega^2,$$

where I_P is the moment of inertia about the axis through P .

From the theorem of parallel axes,

$$I_P = I_{cm} + MR^2. \quad \dots(4)$$

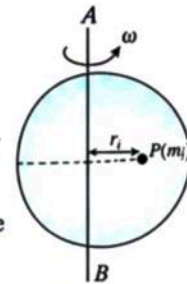


Fig. 5.6.

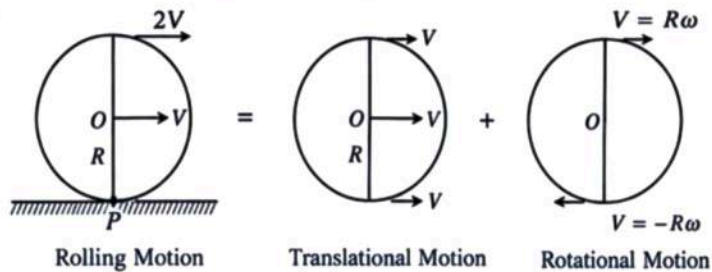


Fig. 5.7.

where I_{cm} is the moment of inertia of the body of mass M about a parallel axis through O .

$$\therefore K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} MR^2 \omega^2 = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} MV^2. \quad \dots(5)$$

If k is the radius of gyration of the body about a parallel axis through O , then $I_{cm} = Mk^2$ and therefore total kinetic energy

$$K = \frac{1}{2} Mk^2 \frac{V^2}{R^2} + \frac{1}{2} MV^2 = \frac{1}{2} MV^2 \left(\frac{k^2}{R^2} + 1 \right). \quad \dots(6)$$

Alternatively, the motion of the cylinder can be considered as the sum of the translational motion with speed V and pure rotational motion about the centre C with angular speed $\omega = V/R$ (fig. 5.7). So that the total kinetic energy is

$$K = \frac{1}{2} MV^2 + \frac{1}{2} I_{cm} \omega^2 = \frac{1}{2} MV^2 \left(1 + \frac{k^2}{R^2} \right). \quad \dots(7)$$

5.6. Motion of a Body Rolling Down an Inclined Plane

If a body rolls down an inclined plane without slipping, it has both translatory and rotatory motions. The body rolls down, hence it suffers vertical descent causing no loss of potential energy. At the same time the body acquires linear and angular velocities and, therefore, it gains kinetic energy of translation and rotation. If there would have been no loss of energy through friction etc., the loss of potential energy must be equal to the gain in kinetic energy.

Let a body of mass M and radius R be rolling down a plane inclined at an angle θ to the horizontal [fig. 5.8]. If the body acquires an angular speed ω and its centre of mass acquires a linear speed V , after covering a distance s along the inclined plane or a vertical height h ($= s \sin \theta$), then

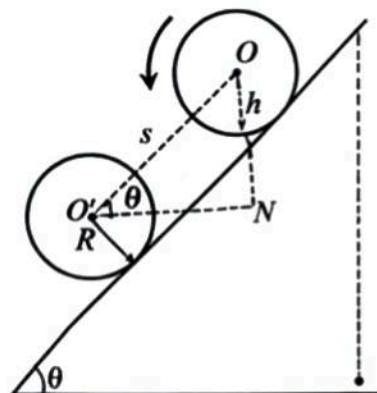


Fig. 5.8.

Loss in potential energy in travelling a vertical height $h = Mgh$.

$$\text{Gain in K.E.} = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} MV^2 = \frac{1}{2} MV^2 \left(\frac{k^2}{R^2} + 1 \right),$$

where k is the radius of gyration about an axis through the centre of mass and parallel to the plane.

According to the law of conservation of energy,

loss in P.E. of the body = gain in K.E. by the body

$$\begin{aligned} \text{i.e.,} \quad Mgh &= \frac{1}{2} MV^2 \left(\frac{k^2}{R^2} + 1 \right) \\ V^2 &= \frac{2gh}{1 + k^2/R^2} = \frac{2gs \sin \theta}{1 + k^2/R^2}. \quad [\because h = s \sin \theta] \quad \dots(1) \end{aligned}$$

If the centre of mass of the body acquires an acceleration a on covering the distance s along the plane after starting from rest, then $V^2 = 2as$. Therefore,

$$2as = \frac{2gs \sin \theta}{1 + k^2/R^2}, \quad \text{So that, } a = \frac{g \sin \theta}{1 + k^2/R^2}. \quad \dots(2)$$

Thus, it is obvious from the above equation that for a given inclination θ , acceleration of the body is inversely proportional to $(1 + k^2/R^2)$. Hence, it follows that lesser the value of k as compared to R the greater is the acceleration of the body rolling down the plane and lesser the time of descent. These effects will be reversed when k is larger. Again, as the expression (2) for acceleration does not contain the mass of the body, the acceleration will be independent of the mass.

Numericals

- Q1.** Identification of the two spheres, one hollow and the other solid but of the same masses (M) and same external radii (R), we allow them to roll down an inclined plane from the same height simultaneously. The moment of inertia of a solid sphere of mass M and radius R about the axis of rotation (diameter) is

$$I = \frac{2}{5} MR^2 = Mk_1^2. \quad \therefore k_1^2 = \frac{2}{5} R^2 \quad \dots(3)$$

where k_1 is the radius of gyration about the axis of rotation.

If the external and internal radii of the hollow sphere are R and R_1 respectively, then its moment of inertia about the axis of rotation is

$$I = \frac{2}{5} M \frac{(R^5 - R_1^5)}{R^3 - R_1^3} = Mk_2^2, \quad \dots(4)$$

where k_2 is the radius of gyration of the hollow sphere about the axis of rotation.

$$k_2^2 = \frac{2}{5} \cdot \frac{R^5 (1 - R_1^5/R^5)}{R^3 (1 - R_1^3/R^3)}, \quad \text{or} \quad k_2^2 = \frac{2}{5} R^2 \frac{1 - (R_1^5/R^5)}{1 - (R_1^3/R^3)}. \quad \dots(5)$$

From eq. (4) and eq. (3), we have

$$\frac{k_2^2}{k_1^2} = \frac{1 - (R_1^5/R^5)}{1 - (R_1^3/R^3)}.$$

But $R > R_1$, hence

$$\frac{R_1^5}{R^5} < \frac{R_1^3}{R^3} \quad \text{or} \quad 1 - \frac{R_1^5}{R^5} > 1 - \frac{R_1^3}{R^3}$$

or

$$\frac{1 - R_1^5/R^5}{1 - R_1^3/R^3} > 1, \quad \therefore k_2^2 > k_1^2. \quad \dots(6)$$

Thus the value of k^2 for a hollow sphere is greater than that for a solid sphere, although the masses and external radii of the two spheres are the same. Hence, it is obvious from the relation (2) that the acceleration of the solid sphere must be greater than that of the hollow one. Therefore, the solid sphere will cover a larger distance on the inclined plane than the hollow sphere in the same interval of time, i.e., if the two spheres are allowed to roll down an inclined plane simultaneously, the solid sphere will reach down the plane first.

Ex. 25. A sphere, a disc, a spherical shell, a ring and a cylinder are allowed to roll down simultaneously on an inclined plane from the same height without slipping. Prove that the sphere reaches down first, then the disc and the cylinder, next the shell and in the last the ring.

Sol. Let M and R be the mass and radius of each body, rolling down an inclined plane at an angle θ with the horizontal. Since the different bodies have different radius of gyration k , they will be accelerated by different amount and hence will require different time intervals in travelling a certain distance on an inclined plane.

(i) Solid sphere : The moment of inertia of a solid sphere about the axis of rotation (or diameter) is given by

$$I = \frac{2}{5} MR^2 = Mk^2, \quad \text{where } k \text{ is the radius of gyration.}$$

$$\therefore \frac{k^2}{R^2} = \frac{2}{5}. \quad \dots(1)$$

$$\text{Acceleration } a_1 = \frac{g \sin \theta}{1 + (k^2/R^2)} = \frac{g \sin \theta}{1 + \frac{2}{5}} = \frac{5}{7} g \sin \theta.$$

(ii) Disc :

$$I = \frac{MR^2}{2} = Mk^2 \quad \text{or} \quad \frac{k^2}{R^2} = \frac{1}{2}, \quad \therefore a_2 = \frac{g \sin \theta}{1 + \frac{1}{2}} = \frac{2}{3} g \sin \theta. \quad \dots(3)$$

(iii) Spherical shell :

$$I = \frac{2}{3} MR^2 = Mk^2 \quad \text{or} \quad \frac{k^2}{R^2} = \frac{2}{3}, \quad \therefore a_3 = \frac{g \sin \theta}{1 + \frac{2}{3}} = \frac{3}{5} g \sin \theta.$$

(iv) Ring :

$$I = \frac{1}{2} MR^2 = Mk^2 \quad \text{or} \quad \frac{k^2}{R^2} = \frac{1}{2}, \quad \therefore a_4 = \frac{g \sin \theta}{1 + \frac{1}{2}} = \frac{1}{2} g \sin \theta.$$

(v) Cylinder :

$$I = \frac{1}{2} MR^2 = Mk^2 \quad \text{or} \quad \frac{k^2}{R^2} = \frac{1}{2}, \quad \therefore a_5 = \frac{g \sin \theta}{1 + \frac{1}{2}} = \frac{2}{3} g \sin \theta = a_2.$$

Obviously, $a_1 > a_2 (= a_5) > a_3 > a_4$, hence, if a sphere, a disc, a cylinder, a spherical shell and a ring are allowed to roll down an inclined plane simultaneously from the same height without slipping, the sphere will reach down first, then the disc and cylinder, then the shell and in the last the ring.

Ex. 26. A solid disc (a) rolls (b) slides from rest down an inclined plane. Neglect friction and compare the velocities in both cases when the disc reaches the bottom of incline.

Sol. Here

The acceleration of the disc, when it rolls down over a plane inclined to an angle θ to the horizontal, is given by

$$a = \frac{g \sin \theta}{1 + k^2/R^2} = \frac{g \sin \theta}{1 + R^2/2R^2} = \frac{2g \sin \theta}{3} \quad \dots(i)$$

If s be the length of the inclined plane, then the velocity V of the disc at the bottom is given by

$$V^2 = 2 \times \frac{2g \sin \theta}{3} \times s, \quad \dots(ii)$$

since the disc starts from rest.

In case, the disc slides through a distance s , the acceleration a' will be given by

$$a' = g \sin \theta \quad \dots(iii)$$

Hence in second case the velocity V' is given by

$$V'^2 = 2g \sin \theta \times s.$$

Dividing (ii) by (i) and taking square root, we get

$$\frac{V}{V'} = \sqrt{\frac{2}{3}}.$$

This is the required comparison of velocities in two cases.

Ex. 27. A uniform sphere of mass 2 kg and radius 10 cm is released from rest on an inclined plane which makes 30° angle with the horizontal. Deduce its

- (i) angular acceleration,
- (ii) linear acceleration along the plane,
- (iii) kinetic energy as it travels 2 metres along the plane.

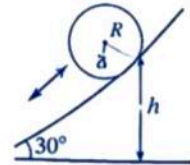
Sol. Linear acceleration along the plane

$$a = \frac{g \sin \theta}{1 + k^2/R^2} = \frac{9.8 \times 1/2}{1 + 2/5} = 3.5 \text{ m/sec}^2. \quad [\because \text{for sphere } k^2 = \frac{2}{5} R^2]$$

Linear velocity, $V = R\alpha$, so $\frac{dV}{dt} = R \frac{d\omega}{dt} = a$, linear acceleration.

$$\therefore \text{Angular acceleration} = \frac{a}{R} = \frac{3.5}{0.1} = 35 \text{ rad/sec}^2.$$

$$\begin{aligned} \text{Gain in K.E.} &= \text{Loss in P.E. in vertical descent } h (= s \sin \theta) \\ &= Mgs \sin \theta = 2 \times 9.8 \times 2 \times \frac{1}{2} = 19.6 \text{ joules.} \end{aligned}$$



Ex. 28. A ball starts from rest and rolls down a 30° inclined plane. How much time would it take to cover 14 metre?

Sol. $a = \frac{g \sin \theta}{1 + k^2/R^2}$

Here, $\frac{k^2}{R^2} = \frac{2}{5}$, $\therefore a = \frac{9.8 \times \sin 30^\circ}{1 + 2/5} = \frac{9.8 \times 0.5}{1.4} = 3.5 \text{ m/sec}^2.$ $[\because I = Mk^2 = \frac{2}{5} MR^2]$

Now, from the equation of motion $s = \frac{1}{2} at^2$

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 14}{3.5}} = 2.8 \text{ sec.}$$