

Elasticity

- A body is said to be rigid, if the relative positions of its constituent particles remains unchanged, when external forces act upon it.
- Elasticity is the property of matter by virtue of which a body regains the original shape or size on the removal of the deforming forces.
- Elastic limit - If the deforming forces are applied on a body upto a certain limit, and the body regains its original shape or size after the removal of the forces, then such a limit is called elastic limit.
- Perfectly Elastic & Plastic Bodies - If the body ^{recovers} completely its original conditions as soon as the deforming forces are removed, it is said to be perfectly elastic, while if it completely retains (remains) its modified form, it is said to be perfectly plastic.
The nearest approach to a perfectly elastic body is Quartz fibre and to a perfectly plastic body is Putty.

Hooke's Law

Within elastic limit, the stress is proportional to the corresponding strain, i.e.,

Stress $\propto$ Strain

or

$$\text{Stress} = E \cdot \text{Strain}$$

$\therefore$

$$E = \frac{\text{Stress}}{\text{Strain}}$$

where; E = Modulus of Elasticity.

Unit & Dimension of E = Dim. of stress

$$= \frac{F}{A} = \frac{m \cdot a}{\text{Area}}$$

$$= \frac{MLT^{-2}}{L^2}$$

$$= [ML^{-1}T^{-2}]$$

{ $\because$ strain is dimensionless }

- Elastic after Effect - when a fine fibre of Quartz is twisted within elastic limit and then released, it comes back to its original form as soon as the twisting force is removed.

Almost, all substances take some time to recover the original condition when deforming forces are removed. This delay in recovering back the original condition, is called elastic after effect for the material.

- Elastic Fatigue - (According to Lord Kelvin) if a wire is loaded repeatedly to a large number of cycles of stresses, it gets tired or ruptured (break) due to gradual fracture of the material and hence loses its strength for some time.

The elastic fatigue may be defined as the loss of the strength of the material for sometime, caused by the repeated stresses in it.

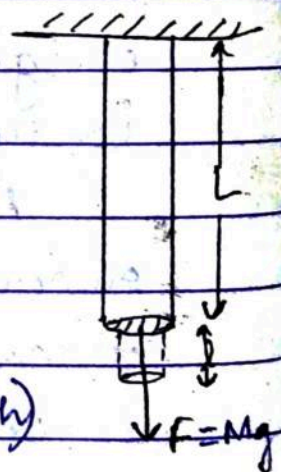
• Elastic Moduli of materials-

i) Young's Modulus: within elastic limit, the ratio of longitudinal stress to the corresponding strain, produced, is called Young's modulus for the material of a body.

Let 'L' be the original (initial) length of a wire of cross-section 'A' and it elongates 'l' by the application of a force 'F'. Then;

$$\text{Longitudinal Stress} = \frac{\text{Force}}{\text{Area}}$$

$$\text{Longitudinal strain} = \frac{l \text{ (change)}}{L \text{ (actual length)}}$$



$$\therefore Y = \frac{F/A}{l/L}$$

If the radius of a wire is 'r', then;

$$A = \pi r^2$$

If 'M' kg mass is suspended at the

lower end of the wire, then;

$$F = M \cdot g$$

Then; Young's modulus,

$$Y = \frac{Mg \cdot L}{\pi r^2 \cdot \Delta L}$$

- Steel OR Iron is more elastic than rubber:- Consider, two wires of steel and rubber of the same length ' L ' and same radius ' r ' and load each of them by same weight ' Mg ', then the change ' ΔL_r ' in the sub length of rubber will be more than the change in length ' ΔL_s ' of steel wire.

[Young modulus of rubber]

Hence;

$$Y_r = \frac{Mg \cdot L}{\pi r^2 \Delta L_r}$$

and;

[of steel]

$$Y_s = \frac{Mg \cdot L}{\pi r^2 \Delta L_s}$$

Therefore;

$$\frac{Y_r}{Y_s} = \frac{\Delta L_s}{\Delta L_r}$$

$$\therefore \Delta L_r > \Delta L_s$$

$\therefore$

$$Y_s > Y_r$$

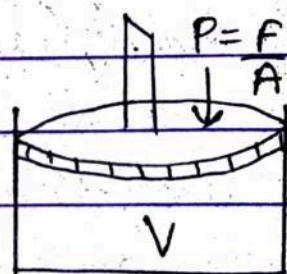
Hence, the value of Young's modulus or coefficient of elasticity of steel or iron is more than that of rubber, i.e.; steel or iron is more elastic than rubber.

ii) Bulk Modulus: Within the limit of elasticity, the ratio of uniform and normal stress on the surface of a body to the volumetric strain is called bulk modulus of the material of the body.

Let 'V' be the original volume of a body and by the application of a pressure 'P', let the volume be changed to '(V - v)'. Then;

$$\text{Stress} = \frac{\text{Normal Force (uniform stress at unit area)}}{\text{Area (unit area)}} = P$$

$$\text{Volume strain} = \frac{-v}{V}$$



$$\therefore \text{Bulk Modulus } K = \frac{\text{stress}}{\text{strain}} = \frac{P}{-v/V}$$

$$\therefore K = -\frac{PV}{v}$$

The reciprocal of Bulk Modulus of a substance, is called ~~as~~ ^{its} compressibility.

$$\text{i.e.; Compressibility} = \frac{1}{K}$$

Note → Isothermal Elasticity ($E_i = P$)
i.e.; the isothermal elasticity of a perfect gas is equal to its original pressure.

- Adiabatic Elasticity ($E_a = \gamma P$) here $\gamma = \frac{C_p}{C_v}$

Then, ratio

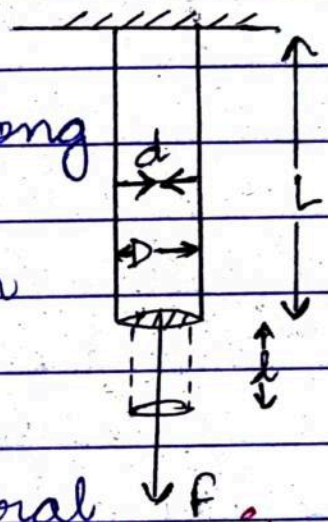
$$\frac{E_a}{E_i} = \gamma$$

Thus, the ratio of two elasticities of a perfect gas is equal to the ratio of its two specific heats.

- Poisson's Ratio

* Lateral strain = $\frac{d}{D}$

When a wire is stretched along its length, it is elongated along with a contraction in its diameter.



Within elastic limit, the lateral strain is proportional to the longitudinal strain, i.e.; the ratio of lateral strain and longitudinal strain is a constant for the material of a body. This constant is known as Poisson's ratio.

Lateral strain $\propto$ Longitudinal strain

$$\text{Lateral strain} = \sigma \cdot \text{Longitudinal strain}$$

$$\therefore \sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

(Poisson's ratio)

If ' β ' and ' α ' are the lateral and longitudinal strains respectively, then

(Poisson's ratio) $\sigma = \frac{\beta}{\alpha}$

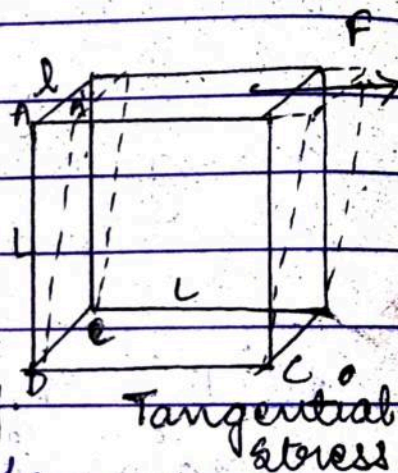
- Poisson's ratio is a pure number and has no units and dimensions since it is the ratio of two strains.
- The limiting values of ' σ ' ~~is~~ are

$$\boxed{-1 < \sigma < 0.5 \left(\frac{1}{2}\right)} \quad \text{(Theoretical limiting value)}$$

Modulus of rigidity or Shear

Modulus

Within the elastic limit, the ratio of tangential stress to the shearing strain is called modulus of rigidity of the material of the body.



In this case, the shape of a body changes but its volume remains unchanged. From figure we have,

$$\text{Shearing strain } \theta = \tan \theta$$

$$= \frac{AA'}{AD} = \frac{BB'}{BC} = \frac{l}{L}$$

$$\text{and, shearing stress} = \frac{F}{l^2 \text{ (area)}}$$

$\therefore$ modulus of rigidity ;

$$\eta = \frac{\text{shearing stress}}{\text{shearing strain}}$$

$$\boxed{\eta = \frac{\theta}{F/l^2}}$$

67. Relations Connecting Various Elastic Constants

These constants are dependent to each other, since any change in the size and shape of a body may be obtained by first changing the size of the body and then changing the shape only. Thus, expressions can be derived, showing inter relations connecting them.

(i) **Relation connecting Y , K and σ [$Y = 3K(1 - 2\sigma)$]:** Let $ABCDEFGH$ represent a cube of unit side (Fig. 6.6). Let us consider a force F , which acts normally and uniformly on each of its six faces in the outward direction. Since each force F is acting on unit area, these forces are representing stresses.

If α is the increase per unit length per unit ^{stress} tension along the direction of the force then the elongation produced in each of the edges, namely AB , BF and BC , will be $F\alpha$. If β is the contraction produced per unit length per unit stress perpendicular to the edges, then contraction produced ~~per unit length~~ to each of the edge, namely AB , BF and BC , will be $F\beta$. Thus the side AB (similarly BF and BC) will increase by $F\alpha$ but simultaneously contract by $F\beta + F\beta$ due to the two other pairs of tensile stresses.

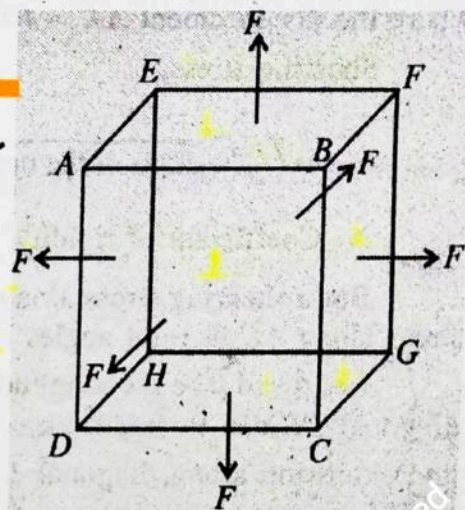


Fig. 6.6.

Hence, the sides of the cube become

$$AE = 1 + F\alpha - F\beta - F\beta = 1 + F(\alpha - 2\beta)$$

$$BF = 1 + F\alpha - F\beta - F\beta = 1 + F(\alpha - 2\beta)$$

$$BC = 1 + F\alpha - F\beta - F\beta = 1 + F(\alpha - 2\beta).$$

Hence, the final volume of the cube is

$$AB \times BF \times BC = [1 + F(\alpha - 2\beta)]^3 = 1 + 3F(\alpha - 2\beta) \text{ [according to Binomial expansion]}$$

The terms containing higher powers of α and β have been neglected since α and β are very small quantities.

As the initial length of a side of the cube is 1, its initial volume = 1

$$\begin{aligned} \therefore \text{Change in volume} &= \text{Final volume} - \text{Initial volume} \\ &= 1 + 3F(\alpha - 2\beta) - 1 = 3F(\alpha - 2\beta). \end{aligned}$$

$$\text{Volume strain} = \frac{3F(\alpha - 2\beta)}{1} = 3F(\alpha - 2\beta)$$

$$\therefore \text{Bulk modulus, } K = \frac{\text{Normal stress}}{\text{Volumetric strain}} = \frac{F}{3F(\alpha - 2\beta)}$$

$$\text{or } K = \frac{1}{3(\alpha - 2\beta)} \text{ or } K = \frac{1/\alpha}{3(1 - 2\beta/\alpha)}$$

But α is the increase per unit length per unit stress, hence clearly

Stress = 1 (as the area of any edge of the cube is unity)

$$\text{and Linear strain} = \alpha/1 = \alpha, \therefore Y = 1/\alpha$$

Also

$\beta/\alpha = \sigma = \text{Poisson's ratio.}$

Therefore,

$$K = \frac{Y}{3(1-2\sigma)} \text{ or } Y = 3K(1-2\sigma) \quad \dots(1)$$

(ii) **Relation connecting Y , η and σ [$\eta = Y/[2(1+\sigma)]$]:** Let $ABCD$ represent the front face of a cube of side L . A tangential force F is applied on its upper face AB and the bottom face DC is fixed. As a result of this force the cube is sheared to $A'B'CD$ through an angle θ [fig. 6.7]. Then, shearing strain is

$$\theta = \tan \theta = \frac{AA'}{AD} = \frac{BB'}{BC} = \frac{l}{L},$$

where the displacement $AA' = BB' = l$.

Shearing stress,

$$T = \frac{F}{\text{Area of the upper face of the cube}} = \frac{F}{L^2}$$

$$\therefore \text{Coefficient of rigidity, } \eta = \frac{T}{\theta}.$$

But a shearing stress along AB is equivalent to a tensile stress along DB and an equal compressive stress along AC at right angles to each other.

Let α and β be the longitudinal and lateral strains per unit stress respectively. Then extension along diagonal DB due to tensile stress = $DB \cdot T \cdot \alpha$

and extension along diagonal DB due to compression stress along $AC = DB \cdot T \cdot \beta$

$$\text{Total extension along } DB = DB \cdot T \cdot (\alpha + \beta) = L\sqrt{2} T (\alpha + \beta).$$

Let us draw a perpendicular BN on DB' . Then increase in the length of diagonal DB is nearly equal to NB' . As θ is very small, therefore $\angle AB'C$ is nearly 90° and $\angle BB'N = 45^\circ$.

$$\text{Thus, } NB' = BB' \cos 45^\circ = \frac{BB'}{\sqrt{2}} = \frac{l}{\sqrt{2}}; \therefore L\sqrt{2} T (\alpha + \beta) = \frac{l}{\sqrt{2}} \text{ or } T \frac{L}{l} = \frac{1}{2(\alpha + \beta)}.$$

$$\text{But } T \cdot \frac{L}{l} = \frac{T}{l/L} = \frac{T}{\theta} = \eta; \therefore \eta = \frac{1}{2(\alpha + \beta)} = \frac{1}{2\alpha(1 + \beta/\alpha)}$$

$$\text{But } \beta/\alpha = \sigma \text{ and } Y = \frac{\text{Stress}}{\text{Longitudinal strain}} = \frac{1}{\alpha} [\because \alpha \text{ is the longitudinal strain per unit stress}]$$

Therefore,

$$\eta = \frac{Y}{2(1+\sigma)}$$

...(2)

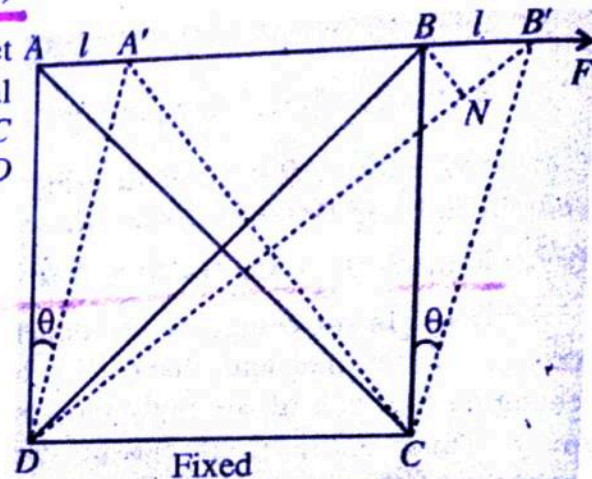


Fig. 6.7.

Relation Connecting Various Elastic Constants

- * Relation connecting Y , K and σ

The relation among Young modulus ' Y '; Bulk modulus ' K ' and Poisson's ratio ' σ ' is given by -

$$K = \frac{Y}{3(1-2\sigma)}$$

- * Relation among Young modulus ' Y '; shearing modulus ' η ' and Poisson's ratio ' σ ' -

$$\eta = \frac{Y}{2(1+\sigma)}$$

- * Relation among Bulk modulus ' K '; shearing modulus ' η ' and Poisson's ratio ' σ ' is given by

As; $K = \frac{Y}{3(1-2\sigma)}$

$$Y = 3K(1-2\sigma) \quad \dots (i)$$

and; $\eta = \frac{Y}{2(1+\sigma)}$

$$Y = 2\eta(1+\sigma) \quad \dots (ii)$$

From eq. (i) and (ii);

$$3K(1-2\sigma) = 2\eta(1+\sigma)$$

or $3K - 6K\sigma = 2\eta + 2\sigma\eta$

$$\sigma = \frac{3K - 2\eta}{6K + 2\eta}$$

- * Relation among Young modulus ' Y '; Bulk modulus ' K ' and shearing modulus ' η ' is -

As; $K = \frac{Y}{3(1-2\sigma)}$

$$\therefore 1 - 2\sigma = \frac{Y}{3K} \dots \text{ci)}$$

and

$$\eta = \frac{Y}{2(1+\sigma)}$$

$$\therefore (2 + 2\sigma) = \frac{Y}{\eta} \dots \text{cii)}$$

From ci) and cii)

$$\frac{3}{2} = \frac{Y}{3K} + \frac{Y}{\eta}$$

Work done Per unit Volume in Deforming a body

In order to deform a body, work has to be done. This energy, so spent is stored up in the body and is called elastic potential energy or internal energy of the strained body or strain energy.

$$\begin{aligned} \text{Strained energy per unit volume} \\ = \frac{1}{2} \times \text{strain} \times \text{stress} \end{aligned}$$

Note: Strain energy or Internal energy of strained wire = $\frac{1}{2} Fl$

Numericals:

Ex. 2. Determine the Poisson's ratio of rubber if due to pulling a rubber thread, its change in volume is negligible in comparison to its change in length.

Sol. If the rubber thread has the length l and radius r , then volume of the thread is

$$V = \pi r^2 l = \text{constant (given)}$$

Now,

$$\delta V = \pi (2r \delta l + r^2 \delta) = 0$$

or

$$2l \delta + r \delta = 0 \text{ or } -\frac{\delta r}{r} = \frac{\delta l}{2l}$$

Hence,

$$\text{Poisson's ratio, } \sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{-\delta r/r}{\delta l/l} = \frac{1}{2} = 0.5.$$

Ex. 3. Young's modulus of a metal is $2 \times 10^{11} \text{ N/m}^2$ and Poisson's ratio is 0.25. Calculate the bulk modulus.

Sol.

$$K = \frac{Y}{3(1-2\sigma)} = \frac{2 \times 10^{11}}{3(1-2 \times 0.25)} = \frac{2 \times 10^{11}}{3 \times 0.5} = 1.33 \times 10^{11} \text{ N/m}^2$$

Ex. 4. Find the Poisson's ratio for steel, if Young's modulus is $2 \times 10^{11} \text{ N/m}^2$ and modulus of rigidity is $8 \times 10^{10} \text{ N/m}^2$.

$$\text{Sol. } \eta = \frac{Y}{2(1+\sigma)}, \text{ whence } \sigma = \frac{Y}{2\eta} - 1 = \frac{2 \times 10^{11}}{2 \times 8 \times 10^{10}} - 1 = 1.25 - 1 = 0.25.$$

✓ **Ex. 5.** A wire of length 5 m and diameter 1 mm is stretched by 0.3 mm by applying a load of 10.0 kgf. Calculate the internal energy of the strained wire. Take $g = 9.8 \text{ m/s}^2$. (Jabalpur 2003)

$$\text{Sol. Strain energy or internal energy of the strained wire} = \frac{1}{2} Fl.$$

$$\text{Here } F = 10 \text{ kgf} = 10 \times 9.8 \text{ N}, l = 0.3 \text{ mm} = 0.3 \times 10^{-3} \text{ m}.$$

$$\therefore \text{Internal energy of the strained wire} = \frac{1}{2} \times (10 \times 9.8) \times (0.3 \times 10^{-3}) = 1.47 \times 10^{-2} \text{ J}.$$

Date _____
Page _____

• Angle of Twist and Angle of shear:

Consider a cylindrical rod or wire of length l and radius r clamped at the upper end and twisted by a couple at the lower free end. By doing so, each circular cross-section of the rod rotates through an angle. This angle is called the angle of twist and is proportional to the distance of cross section from the clamped end.

If the angle of twist at the free end is θ , then

$$BB' = r\theta = l\phi \dots (1) \therefore \text{Angle} = \frac{\text{Arc}}{\text{Radius}}$$

where ϕ is the angle of shear.

Further consider a cross-section at a distance l' from the fixed end and let the angle of twist be θ' , then

$$cc' = r\theta' = l'\phi \dots (2)$$

Dividing eqs. (2) by eqs (1), we get

$$\frac{r\theta'}{r\theta} = \frac{l'\phi}{l\phi}$$

$$\text{or. } \frac{\theta'}{\theta} = \frac{l'}{l}$$

Hence, For $l' < l$,
 $\theta' < \theta$.

At the fixed end, $l = 0$ and hence $\theta = 0$.

- The Angle of shear is a constant for any cylindrical surface, goes on decreasing as the radius of the surface decreases.

Now, consider the shearing of any two co-axial cylindrical layers of radius r and r' . The angle of twist θ at the lower free end is a constant.

From Figure;

$$L\phi = r\theta \quad (\text{Outer cylindrical layer})$$

$$\text{and } L\phi' = r'\theta \quad (\text{Inner cylindrical layer})$$

$$\text{As } r > r'$$

$$\therefore \phi > \phi'$$

$$\left\{ \begin{array}{l} \therefore L\phi = r\theta \\ L\phi' = r'\theta \end{array} \right.$$

$$\text{so, } \frac{\phi}{\phi'} = \frac{r}{r'}$$

$$\text{At the axis, } r = 0$$

$$\therefore \phi = 0$$

- Angle of shear is a maximum on the surface of the cylinder and minimum on the axis of the cylinder.

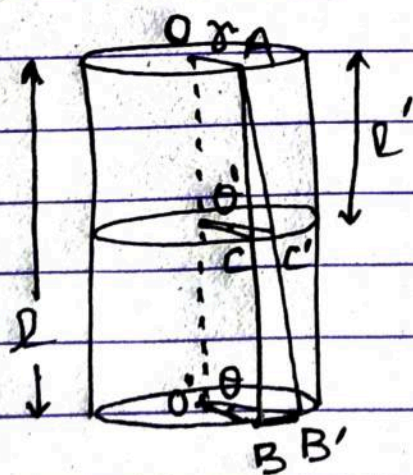
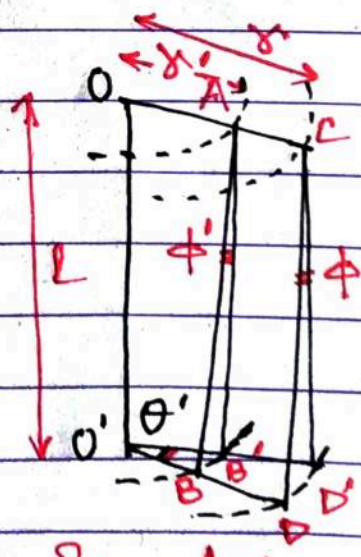


Figure.



$$\text{Angle} = \frac{\text{Arc}}{\text{radius}}$$

Twisting Couple on a Cylindrical rod or wire

Consider, a cylindrical rod of length ' l ' and radius ' r '. The upper end of the rod is kept fixed and a twisting couple is applied at its lower end in a plane perpendicular to its length. If ' θ ' is the angle of twist at the lower end of the rod, then -

$$\text{Twisting couple on the rod} = \frac{\eta \pi r^4}{2l} \cdot \theta$$

where; $\eta \rightarrow$ shearing modulus (or modulus of rigidity) of the material of the rod.

If ' $\theta = 1$ ', then twisting couple per unit twist ' C ' is -

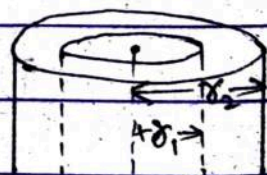
$$C = \frac{\eta \pi r^4}{2l}$$

[Restoring couple per unit twist]

This ' C ' is called the Torsional rigidity of the material of the wire.

Twisting Couple for a hollow cylindrical rod

If the cylindrical rod is hollow, having internal and external radii respectively, then couple required to twist the rod through an angle ' θ ' at its lower end is,



$$\text{Twisting couple} = \frac{\eta \pi (r_2^4 - r_1^4)}{2l} \cdot \theta \quad [r_2 > r_1]$$

Hence, the couple required to twist the hollow rod through 1 radian, i.e.

Torsional rigidity is given by -

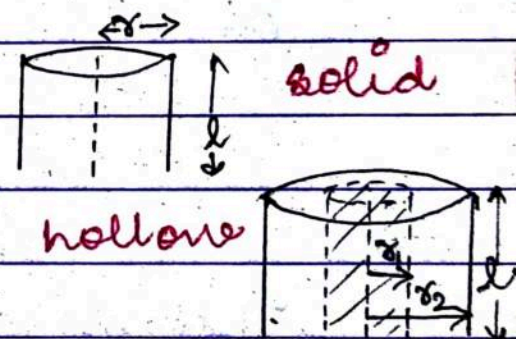
$$C' = \frac{\eta \pi (r_2^4 - r_1^4)}{2l}$$

Comparison of Torsional Rigidities for hollow and solid cylindrical rods

If we have two cylinders, one hollow and other solid, of the same material, length and mass, the couples required to twist them through one radian are ' C' ' and ' C ' respectively.

$$C = \frac{\eta \pi r^4}{2l}$$

$$C' = \frac{\eta \pi (r_2^4 - r_1^4)}{2l}$$



$$\text{Then, } \frac{C'}{C} = \frac{r_2^4 - r_1^4}{r^4} = \frac{(r_2^2 + r_1^2)(r_2^2 - r_1^2)}{r^4} \dots (i)$$

But, the two cylinders ^{are} have the same mass and are made of the same material.

$$(\pi r_2^2 l - \pi r_1^2 l) \rho = \pi r^2 l \rho$$
$$\therefore (r_2^2 - r_1^2) = r^2 \dots (ii)$$

Therefore,

$$\frac{C'}{C} = \frac{(r_2^2 + r_1^2)(r_2^2 - r_1^2)}{r^4} \quad [\because r_2^2 - r_1^2 = r^2]$$

$$\boxed{\frac{C'}{C} = \frac{r_1^2 + r_2^2}{r^2}} \quad \dots (iii)$$

As, $r_2^2 - r_1^2 = r^2$ [from eq. (ii)]

or, $r_2^2 = r^2 + r_1^2$

$\therefore r_1^2 + r_2^2 = r^2 + 2r_1^2$

Hence, $(r_1^2 + r_2^2) > r^2$

$\therefore \frac{r_1^2 + r_2^2}{r^2} > 1$

Thus,

$$\frac{C'}{C} = \frac{r_1^2 + r_2^2}{r^2} > 1$$

or $\frac{C'}{C} > 1$

$\therefore \boxed{C' > C}$

This means that if two cylinders (one hollow and other solid) of same length, mass and material are given. Then, to twist the cylinders through the same angle, a greater couple is required to twist the hollow cylinder. Thus, a hollow shaft (axle) can transmit larger torques (couples) for the same angle of twist.

Work Done in Twisting a Wire or Rod

Let a wire or rod of length l and radius r be fixed at the upper end. At the lower end of the wire, a couple is applied so that a twist of an angle θ is produced at this end.

If C be the couple per unit twist of the wire then the couple, required to twist the wire through an angle θ , will be $C\theta$. Now, the work done in twisting the wire further through a small angle $d\theta$ is given by

$$dW = C\theta \cdot d\theta$$

Hence, the total work done in twisting the wire through the angle θ is

$$W = \int_0^\theta C\theta \cdot d\theta = \frac{1}{2} C\theta^2 = \frac{\eta\pi r^4 \theta^2}{4l} \quad \left[\because C = \frac{\eta\pi r^4}{2l} \right] \dots(10)$$

The energy, spent in doing this work, is stored up in the wire and is known as **strain energy**.

Bending of a beam

1 Beam - A rod (or bar) of uniform rectangular or circular cross-section, whose length is very large in comparison to its thickness or radius is called a beam.

2 Neutral Surface - When a beam is clamped horizontally at one end and loaded at the other, it undergoes bending. The filaments (or fibres) of outward side are lengthened and subjected to tension while those of the inner side are subjected to compression and shortened.

In between these two portions, there is a layer or surface in which the filaments are neither elongated nor shortened. Such a surface is called neutral surface.

3 Bending Moment - The quantity ' $\frac{YI}{R}$ ' is the restoring couple or the bending moment of a beam.

i.e.,
$$\text{Bending moment} = \frac{YI}{R}$$

The quantity ' YI ' is known as Flexural rigidity.

where, $R \rightarrow$ radius of curvature of the neutral surface.

and,

- In case of a beam of rectangular cross section of breadth ' b ' and depth ' d ', the moment of inertia of the beam is,

$$I = \frac{bd^3}{12}$$

- In case of a cylindrical rod of radius ' r '; the moment of inertia of the beam is

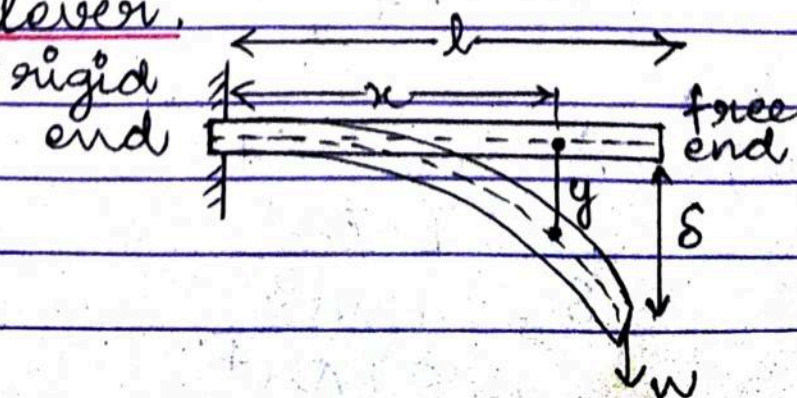
$$I = \frac{\pi r^4}{4}$$

- In case, the rod is hollow, having r_1 and ' r_2 ' as the internal and external radii, then

$$I = \frac{\pi (r_2^4 - r_1^4)}{4}$$

Cantilever

A horizontal beam, fixed at one end and loaded at the free end, is called a Cantilever.



Let a beam of length 'l' be fixed at its one end and loaded with a load 'W' at the other end.

The expression for the depression 'y' at a distance 'x' from the fixed end is -

$$y = \frac{W}{YI} \left(\frac{lx^2}{2} - \frac{x^3}{6} \right) \quad *$$

At the free end; 'x=l' and 'y=δ'; then, the expression for depression is -

$$\delta = \frac{W}{YI} \left(\frac{l^3}{2} - \frac{l^3}{6} \right)$$

$$\therefore \delta = \frac{Wl^3}{3YI}$$

case-I For a beam of rectangular cross-section, of breadth 'b' and depth 'd'; $I = \frac{bd^3}{12}$

Therefore; $\delta = \frac{Wl^3 \times 12}{3Y \times bd^3}$

$$\delta = \frac{4Wl^3}{bd^3Y}$$

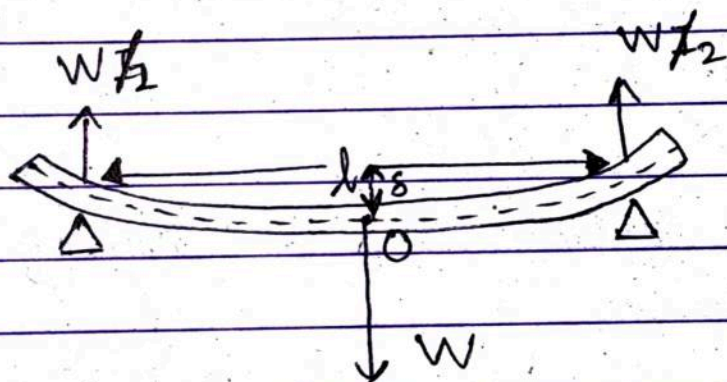
case-II For a beam of circular cross-section of radius 'r'; $I = \frac{\pi r^4}{4}$

Therefore; $\delta = \frac{Wl^3 \times 4}{3Y \times \pi r^4}$

$$\delta = \frac{4Wl^3}{3Y\pi r^4}$$

Beam Supported at its ends and loaded in the middle.

Consider a beam supported on the two edges at a distance 'l' apart in a horizontal plane. Let, it be loaded at its middle point. The reaction of each edge is equal to ' $\frac{W}{2}$ ' acting upwards



The depression at the middle 'O' of the beam is

$$\delta = \frac{Wl^3}{48YI}$$

- If 'b' and 'd' be the breadth and depth of a beam; then, $I = \frac{bd^3}{12}$

$$\therefore \delta = \frac{Wl^3 \times 12}{48Y \times bd^3}$$

So, $\delta = \frac{Wl^3}{4Ybd^3}$

Girders are always I shaped.

Date _____

Page _____

86

The use of this equation is made in the construction of girders. Girders are I shaped.

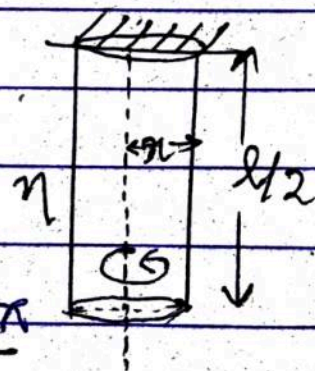
Note: For given load, $F = mg$ and same cross-section, the ratio of depressions for a rod of square and circular cross-sections is —
 $= 3 : \pi$

- Torsional couple required to twist a cylinder of length $l/2$, radius r and modulus of rigidity η by one turn will be —

Twisting couple — $\frac{\eta \pi r^4 \cdot \theta}{2l}$

$$= \frac{\eta \pi r^4}{2 \times \frac{l}{2}} \cdot 2\pi$$

$$= \frac{\eta 2\pi^2 r^4}{l}$$



● Ex. 3. A cantilever of length l and uniform cross-section shows a depression of 2 cm at the loaded end. What will be the depression at distance $l/4$, $l/2$ and $3l/4$ from the fixed end ?

Sol.

$$y = \frac{Mgx^2}{2YI} \left(1 - \frac{x}{3}\right)$$

When $x = l$, $y = 0.02$ m,

$$\therefore 0.02 = \frac{Mgl^2}{2YI} \left(1 - \frac{1}{3}\right) = \frac{Mgl^2}{3YI}$$

✓ When $x = l/4$,

$$y_1 = \frac{Mg}{2YI} \left(\frac{l}{4}\right)^2 \left(1 - \frac{1}{12}\right) = \frac{Mgl^2}{32YI} \cdot \frac{11}{12} = \frac{11}{128} \cdot \frac{Mgl^2}{3YI}$$

$$= \frac{11}{128} \times 0.02 = 1.72 \times 10^{-3} \text{ m.}$$

✓ When $x = l/2$,

$$y_2 = \frac{Mg}{2Yl} \left(\frac{l}{4}\right)^2 \left(1 - \frac{l}{6}\right) = \frac{Mgl^3}{8Yl} \cdot \frac{5}{6} = \frac{5}{16} \cdot \frac{Mgl^3}{3Yl}$$

$$= \frac{5}{16} \times 0.02 = 6.2 \times 10^{-3} \text{ m.}$$

✓ When $x = \frac{3l}{4}$

$$y_3 = \frac{Mg}{2Yl} \left(\frac{3l}{4}\right)^2 \left(1 - \frac{1}{4}\right) = \frac{Mgl^3}{2Yl} \cdot \frac{27}{64} = \frac{Mgl^3}{3Yl} \cdot \frac{81}{128}$$

$$= \frac{81}{128} \times 0.02 = 1.27 \times 10^{-2} \text{ m.}$$

Ex. 7. Compare loads required to produce equal depression for two beams, made of the same material and having the same length and weight with the only difference that while one has circular cross-section, the cross-section of the other is square.

Sol. The depression δ at the middle point of a beam of length l of circular cross-section of radius r , is given by

$$\delta = \frac{W_1 l^3}{12Y\pi r^4}, \text{ where } W_1 \text{ is the weight loaded.} \quad \text{---(i)}$$

If in case of rectangular cross-section beam the weight loaded be W_2 , then the depression δ , is given by

$$\delta' = \frac{W_2 l^3}{4b d^3 Y}$$

since the two beams have the same length and are made of same material.

But for square cross-section $b = d$

$$\therefore \delta' = \frac{W_2 l^3}{4b^4 Y} \quad \text{---(ii)}$$

The two beams have equal weights and made of the same material, therefore their masses and hence volume must be equal, i.e., $\pi r^2 l = b^2 l$ or $b^2 = \pi r^2$

Substituting the value of b^2 in (ii), we get

$$\delta' = \frac{W_2 l^3}{4\pi^2 r^4 Y} \quad \text{---(iii)}$$

But depression is the same in both cases, i.e., $\delta = \delta'$. Therefore equating (i) and (iii), we get

$$\frac{W_1 l^3}{12Y\pi r^4} = \frac{W_2 l^3}{4\pi^2 r^4 Y} \text{ or } \frac{W_1}{W_2} = \frac{3}{\pi}, \text{ which is the required ratio.}$$