

Macrostate and Microstate:

Macrostate:

Consider 4 distinguishable particles and we wish to distribute them into two exactly similar compartments in an open box. Let the particles be a, b, c and d. When any particle is thrown into the box, it must fall into one of the two compartments. Since the compartments are alike, the particles have the same a priori probability of going into either of them and will be $\frac{1}{2}$. The possible ways in which 4 particles can be distributed in two compartments is shown in figure:

compartment	Number of particles				
	Distribution 1	Distribution 2	Distribution 3	Distribution 4	Distribution 5
1	0	1	2	3	4
2	4	3	2	1	0

Thus, there are 5 different distributions (0,4), (1,3), (2,2), (3,1) and (4,0). **Each compartment wise distribution of a system of particles is known as a macrostate.** In general, for a system of n particles to be distributed in two similar compartments, the total number of macrostates for n particles is $(n+1)$.

Microstate:

Since the particles are distinguishable, the number of different possible arrangements in each compartment is shown in figure:

Macrostate	Possible arrangement		Number of microstates W (Thermodynamic probability)
	Compartment 1	Compartment 2	
0,4	0	abcd	1
1,3	a b c d	bcd cda dab abc	4
2,2	ab ac ad bc bd cd	cd bd bc ad ac ab	6
3,1	bcd cda dab abc	a b c d	4
4,0	abcd	0	1

Each distinct arrangement is known as the microstate of the system. In general, for a system of n particles, the total number of microstates are 2^n .

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***Thermodynamic Probability:**

The thermodynamic probability of a particular macrostate is defined as the number of microstates corresponding to that macrostate. The number of microstates corresponding to a given macrostate is equal to the number of meaningful arrangements.

For the case of n particles and two compartments, if r is the number of particles in the compartment no.1 and the remaining $(n-r)$ are in compartment no.2, then

$$\begin{aligned}\text{No. of meaningful arrangements} &= \frac{n!}{r!(n-r)!} \\ &= {}^nC_r\end{aligned}$$

Therefore, the number of microstates in a macrostate $(r, n-r)$ or thermodynamic probability

$$W_{(r, n-r)} = \frac{n!}{r!(n-r)!} = {}^nC_r \quad \dots\dots\dots(1)$$

***Probability:**

The probability (of occurrence) of a macrostate is defined as the ratio of the number of microstates (i.e., thermodynamic probability W) in it to the total number of possible microstates of the system.

Thus,

$$P_{\text{macro}} = \frac{\text{No. of microstates in the macrostate}}{\text{Total no. of microstates of the system}} \quad \dots\dots\dots(2)$$

The total number of ways of arranging n distinguishable particles in c numbered compartments $= c^n$. If there are only 2 compartments or cells, then the total no. of microstates of the system $= 2^n$.

$$\text{Therefore, } P_{\text{macro}} = \frac{W}{c^n} = \frac{W}{2^n} \quad \dots\dots\dots(3)$$

Substituting for W from equation (1),

$$P_{\text{macro}} = \frac{n!}{r!(n-r)!} \times \frac{1}{2^n} = \frac{1}{2^n} \times {}^nC_r$$

This gives the probability (of occurrence) of macrostate $(r, n-r)$.

***Combination Possessing Maximum Probability:**

The value of ${}^nC_r$ is maximum (greatest) when $r = \frac{n}{2}$. Then the maximum value of probability is given by

$$P_{\text{max}} = \frac{1}{2^n} {}^nC_{n/2} = \frac{1}{2^n} \frac{n!}{\frac{n!}{2} \frac{n!}{2}} = \frac{n!}{\left(\frac{n!}{2}\right)^2} \frac{1}{2^n}$$

***Combination possessing minimum probability:**

The value of ${}^nC_r$ is minimum, if r is equal to either zero or n .

If $r=0$, the probability is

$$\begin{aligned}P_{\min} &= \frac{n!}{r!(n-r)!} \times \frac{1}{2^n} \\&= \frac{n!}{0!(n-0)!} \frac{1}{2^n} \\&= \frac{1}{2^n}\end{aligned}$$

since $(0!=1)$

EXAMPLE

Calculate the probability that in tossing a coin 5 times, we get 3 heads and 2 tails.

(Punjab 1998)

SOLUTION. Tossing a coin 5 times is equivalent to tossing 5 similar coins simultaneously. Hence $n = 5$. We wish to get 3 heads and 2 tails i.e.,

$$r = 3 \text{ and } (n - r) = 2$$

The probability of distribution $(r, n - r)$ is given by

$$\begin{aligned} P_{(r, n-r)} &= \frac{1}{2^n} \cdot {}^n C_r = \frac{1}{2^n} \cdot \frac{n!}{r!(n-r)!} \\ &= \frac{1}{2^5} \times \frac{5!}{3!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{32 \times 3 \times 2 \times 1 \times 2 \times 1} \\ &= \frac{10}{32} = \frac{5}{16} \end{aligned}$$

EXAMPLE

Calculate the probability that in tossing a coin 10 times, we get (i) all heads (ii) 5 heads & 5 tails and (iii) 3 heads & 7 tails. (iv) 7 heads & 3 tails.

(Meerut 2004)

SOLUTION. The probability of distribution $(n_1, n - n_1)$ is given by

$$P_{(r, n-r)} = \frac{1}{2^n} \cdot \frac{n!}{r!(n-r)!}$$

(i) Tossing a coin 10 times is equivalent to tossing 10 similar coins simultaneously

Here, $n = 10$

We wish to have all heads

$$\therefore r = 10 \text{ and } (n - r) = 0$$

$$\text{Hence, } P_{(10, 0)} = \frac{1}{2^{10}} \cdot {}^{10} C_{10} = \frac{1}{2^{10}} \times \frac{10!}{10!0!} = \frac{10!}{10!} \quad (\because 0! = 1)$$

$$\therefore P_{(10, 0)} = \frac{1}{2^{10}} = \frac{1}{1024} = 0.00098$$

(ii) We wish to have 5 heads and 5 tails

$$\therefore r = 5 \text{ and } (n - r) = 5$$

$$\begin{aligned} \text{Hence } P_{(5, 5)} &= \frac{1}{2^{10}} \cdot {}^{10} C_5 = \frac{1}{2^{10}} \cdot \frac{10!}{5!5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1} \times \frac{1}{1024} \\ &= \frac{252}{1024} = 0.24609 \end{aligned}$$

(iii) We wish to have 3 heads and 7 tails

$$\therefore r = 3 \text{ and } (n - r) = 7$$

$$\begin{aligned} \text{Hence, } P_{(3, 7)} &= \frac{1}{2^{10}} \cdot {}^{10} C_3 = \frac{1}{1024} \times \frac{10!}{3!7!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times \frac{1}{1024} \end{aligned}$$

$$= \frac{120}{1024} = 0.11719$$

(iv) We wish to have 7 heads and 3 tails

Here

$$r = 7 \text{ and } (n - r) = 3$$

$$P_{(7,3)} = \frac{1}{2^{10}} \cdot {}^{10}C_7 = \frac{1}{2^{10}} \cdot \frac{10!}{7!3!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} \times \frac{1}{1024}$$

$$= \frac{120}{1024} = 0.11719$$

Thus, the probability of distribution (7, 3) is equal to the probability of distribution (3, 7)