

1. Constraints

- ✓ Constraints are restrictions imposed on the position or motion of a system because of geometrical conditions.

2. Examples

- 1 The beads of an abacus are constrained to one-dimensional motion by the supporting wires.
- 2 Gas molecules within a container are constrained by the walls of the vessel to move only inside the container.
- 3 The motion of rigid bodies is always such that the distance between any two particles remains unchanged.
- 4 A particle placed on the surface of a solid sphere is restricted by the constraint so that it can only move on the surface or in the region exterior to the sphere.
- 5 The motion of point mass of a simple pendulum is restricted since the point mass always remains at a constant distance from the point of suspension.

Types of Constraints

► Holonomic and non-holonomic constraints

- If the constraints can be expressed as equations connecting the co-ordinates of the particles (and possibly time) in the form

$$f(r_1, r_2, r_3, \dots, r_n, t) = 0$$

then the constraints are said to be **holonomic**.

Examples. (1) The constraints involved in the motion of rigid bodies in which the distance between any two particular points is always fixed, are *holonomic* since the conditions of constraints are expressed as

$$(r_1 - r_2)^2 - c_{12}^2 = 0.$$

- (2) The constraints involved when a particle is restricted to move along a curve or surface are *holonomic*. Here the equation defining the curve or surface is the equation of constraint.
- (3) The constraints involved in the motion of the point mass of a simple pendulum are *holonomic*.

Let

r = position vector of the point mass

θ = position vector of the point of suspension

l = length of the string.

The equation of the constraint may be written as

$$(r - a)^2 - l^2 = 0.$$

- If the constraints cannot be expressed in the form of Eq. (1), they are called *non-holonomic constraints*.

Examples. (1) The constraints involved in the motion of the particle placed on the surface of a solid sphere are non-holonomic. The conditions of constraints in this case are expressed as

$$r^2 - a^2 \geq 0,$$

where a is the radius of sphere. This is an inequality and hence not in the form of Eq. (1).

(2) The walls of the gas vessel constitute a non-holonomic constraint.

(3) Another example of a non-holonomic constraint is the condition that the velocity of the point of contact of a sphere, rolling without slipping along a stationary rough surface, is equal to zero.

(ii) *Scleronomic and Rheonomic Constraints.* If the constraints are independent of time, they are called *scleronomic*.

The constraint in the case of rigid body motion is *scleronomic*. If the constraints are explicitly dependent on time, they are called *rheonomic*. A bead sliding on a moving wire is an example of *rheonomic* constraint.

Principle of virtual work

Let us imagine two possible configurations of a system of particles at any instant of time which are consistent with the forces and constraints. If the system of particles goes from one configuration to the other, each particle of the system is imagined to be displaced by an infinitesimal vector $\delta \mathbf{r}_i$ and is called the virtual displacement to distinguish from the actual displacement $d\mathbf{r}_i$ which occurs in a time interval dt during which the forces and constraints may be changing.

* In order for a system to be in equilibrium, the resultant force acting on each particle of the system must be zero, i.e., $\mathbf{F}_i = 0$, where $\mathbf{F}_i$ is the force acting on i th particle of the system. Then clearly, the dot product of force $\mathbf{F}_i$ and the virtual displacement $\delta \mathbf{r}_i$ due to force $\mathbf{F}_i$, which is the virtual work must vanish.

i.e., $\mathbf{F}_i \cdot \delta \mathbf{r}_i = \text{virtual work} = 0.$

Summing over all the particles of the system, we have

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0, \quad \dots (4) \quad 1$$

— since the sum of the vanishing products over all the particles of the system is also zero.

✓ If the constraints are present, then the force $\mathbf{F}_i$ acting on the i th particle is written as the sum of constraint force and actual force, i.e.,

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i. \quad \dots (5) \quad 2$$

$\mathbf{F}_i^a$ being the actual force and $\mathbf{f}_i$ being the force of constraint acting on i th particle.

Putting value of $\mathbf{F}_i$ from equation (5) in equation (4), we have

$$\sum_i (\mathbf{F}_i^a + \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0$$

or

$$\sum_i (\mathbf{F}_i^a + \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0 \quad \dots (6) \quad 3$$

Now restricting ourselves to the system where the virtual work done by the forces of constraints, is zero i.e.,

$$\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

Now equation (6) reduces to

$$\sum F_i^a \cdot \delta r_i = 0. \quad (7) 4$$

This is the condition for equilibrium of a system according to which, "A system of particles is in equilibrium only if the total virtual work of the actual or applied forces is zero." This is known as the principle of virtual work.

D' Alembert's Principle

The principle of virtual work stated above is applicable to the statics of a system of particles : but it can be related to obtain a similar principle of dynamics.

According to Newton's second law of motion, the force is defined as rate change of momentum, i.e.,

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt} = \dot{\mathbf{p}}_i.$$

where $\mathbf{p}_i$ is the momentum of i th particle due to force $\mathbf{F}_i$ acting on i th particle

$$(\mathbf{F}_i - \dot{\mathbf{p}}_i) = 0 \quad (8) 5$$

According to the above equation a moving system of particles can be considered to be in equilibrium under the force $(\mathbf{F}_i - \dot{\mathbf{p}}_i)$ i.e., the actual applied force $\mathbf{F}_i$ plus an additional force $-\dot{\mathbf{p}}_i$ which is known as reversed effective force on i th particle.

Now replacing $\mathbf{F}_i$ of equation (4) by $(\mathbf{F}_i - \dot{\mathbf{p}}_i)$ we have

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0. \quad (9) 6$$

If the forces of constraints are present, from equation (5) we have

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i. \quad (10) 7$$

Putting value of $\mathbf{F}_i$ from equation (10) in equation (9), we get

$$\sum_i \{(\mathbf{F}_i^a + \mathbf{f}_i) - \dot{\mathbf{p}}_i\} \cdot \delta \mathbf{r}_i = 0$$

or

$$\sum_i (\mathbf{F}_i^a - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0. \quad (11) 8$$

Again, restricting ourselves to the cases where the virtual work done by the forces of constraints vanishes, i.e.,

$$\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

34.
Equation (11) reduces to

$$\sum_i (\mathbf{F}_i^a - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0,$$

which is called *D' Alembert's principle.*

The Lagrangian Function

The Lagrangian function is the difference between the kinetic energy (T) and potential energy (V) of a system. It is a function of generalised co-ordinates, generalised velocities and time.

$$L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k, t) - V(q, t)$$

* For a conservative system:

The potential energy and Lagrangian function of a conservative system do not depend explicitly on time.

$$\therefore L(q_k, \dot{q}_k) = T(q_k, \dot{q}_k) - V(q)$$

[For a conservative system]

* Lagrangian Equation

The Lagrangian equation for a conservative system is given by:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \quad (\text{No derivation})$$

Lagrangian formulation of conservation theorems.

For the purpose let us first define *generalised momentum* and *cyclic co-ordinates*.

- **Generalised momentum.**

Consider a conservative system, *i.e.*, a system in which the forces are derivable from a potential function V dependent on position only ; then

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial (T - V)}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k}.$$

$$\begin{aligned}
 &= \frac{\partial T}{\partial \dot{q}_k} \quad (\text{because potential } V \text{ depends only on position or } \partial V / \partial q_k = 0) \\
 &= \frac{\partial}{\partial \dot{q}_k} \sum_k \frac{1}{2} m_k \dot{q}_k^2 \\
 &= m_k \ddot{q}_k = p_k.
 \end{aligned}$$

p_k is called the *generalised momentum* associated with generalised co-ordinate q_k .

Thus the generalised momentum corresponding to the generalised co-ordinate q_k is defined as the quantity $\frac{\partial L}{\partial \dot{q}_k}$ and is represented by p_k

i.e.,
$$p_k = \frac{\partial L}{\partial \dot{q}_k}. \quad \text{--- (1)}$$

Cyclic or ignorable co-ordinates,

If the Lagrangian of a system is not the function of a given co-ordinate q_k , then the co-ordinate q_k is said to be cyclic or ignorable.

In such case
$$\frac{\partial L}{\partial q_k} = 0. \quad \dots (2)$$

(i) Conservation theorem for generalised momentum. According to Lagrange's equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0, \quad \text{--- (3)}$$

If the generalised co-ordinate q_k is cyclic or ignorable, then we have

$$\frac{\partial L}{\partial q_k} = 0 \quad [\text{from equation (2)}]$$

With this substitution equation (3) becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0$$

or
$$\frac{\partial L}{\partial \dot{q}_k} = \text{constant}$$

i.e.,
$$p_k = \text{constant} \quad [\text{from equation (1)}].$$

This gives the conservation theorem for generalised momentum which states that the generalised momentum corresponding to a cyclic or ignorable co-ordinate is conserved.

(ii) Conservation theorem for energy.

Consider a conservative system, i.e., a system in which the forces are derivable from a potential function V dependent on position only. The Lagrangian L is given by

$$L = L(q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots).$$

Let us assume that L does not depend upon time explicitly. Then

$$L = L(q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots).$$

The total time derivative of L is given by

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial \dot{q}_1} \dot{q}_1 + \frac{\partial L}{\partial \dot{q}_2} \dot{q}_2 + \dots + \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k + \dots + \frac{\partial L}{\partial \dot{q}_1} \dot{q}_1 + \frac{\partial L}{\partial \dot{q}_2} \dot{q}_2 \\ &\quad + \dots + \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k + \dots \\ &= \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k. \end{aligned} \quad (4)$$

Lagrange's equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0,$$

or

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = \frac{\partial L}{\partial q_k}.$$

With this substitution equation (4) becomes

$$\begin{aligned} \frac{dL}{dt} &= \sum_k \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k, \\ &= \sum_k \frac{d}{dt} \left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \right), \end{aligned}$$

or

$$\frac{d}{dt} \left(\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \right) = 0.$$

Thereby indicating that $\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{constant}$.

But

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_k} &= \frac{\partial}{\partial \dot{q}_k} (T - V) = \frac{\partial T}{\partial \dot{q}_k} \\ &= \frac{\partial}{\partial \dot{q}_k} \left(\sum_k \frac{1}{2} m_k \dot{q}_k^2 \right) \end{aligned}$$

because $T = \sum_k \frac{1}{2} m_k \dot{q}_k^2$, where m_k is the mass of k th particle.

$$\therefore \frac{\partial L}{\partial \dot{q}_k} = m_k \dot{q}_k.$$

Substituting the value of $\frac{\partial L}{\partial \dot{q}_k}$ from above equation in (5), we get

$$\sum_k m_k \dot{q}_k^2 - L = \text{constant}$$

or

$$2T - L = \text{constant}$$

or

$$2T - (T - V) = \text{constant}$$

Therefore $T + V = \text{total energy} = \text{constant}$

Thus the energy conservation theorem states that, if the Lagrangian function does not contain the time explicitly, the total energy of the conservative system is conserved.

Application of Lagrange's Equations

(b) **Simple pendulum:** Let the mass of the pendulum bob be m . The angle θ between rest position (OA) and deflected position (OB) is chosen as the generalized coordinate (Fig. 42.9). Let l be the length of the string. Then

$$\text{K.A. of the bob} = \frac{1}{2}mv^2$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

P.E. of the bob is

$$V = mg \, CA = mg(OA - OC) = mg(l - l \cos \theta) = mgl(1 - \cos \theta)$$

The Lagrangian function L is given by

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos \theta) \quad \dots(1)$$

The Lagrangian equation in the generalized coordinate θ is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad \dots(2)$$

From Eq. (1), $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = -mgl \sin \theta$

Substituting these values in Eq. (2), we get

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl \sin \theta = 0$$

or $ml^2\ddot{\theta} + mgl \sin \theta = 0$

or $\ddot{\theta} + \frac{g}{l} \sin \theta = 0$

If the amplitude of motion is small, $\sin \theta \approx \theta$. Then

$$\ddot{\theta} + \frac{g}{l} \theta = 0 \quad \dots(3)$$

Eq. (3) represents a S.H.M. with period $T = 2\pi \sqrt{l/g}$

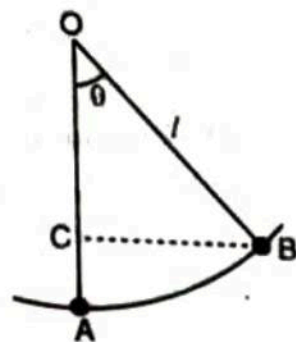


Fig. 42.9

13.4 (b) The Hamiltonian Function H.

The Lagrangian, in general is the function of

$q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots$ and t , i.e.,

$$L = L(q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots, t) = L(q_k, \dot{q}_k, t)$$

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial L}{\partial t} \quad \dots(1)$$

The Lagrangian equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

or

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \quad \dots(2)$$

With this substitution equation (1) becomes

$$\frac{dL}{dt} = \sum_k \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial L}{\partial t}$$

$$= \sum_k \frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k \right\} + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt} \left\{ L - \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k \right\} = \frac{\partial L}{\partial t} \quad \dots(3)$$

But

$$\frac{\partial L}{\partial \dot{q}_k} = p_k = \text{generalised momentum}$$

The equation (3) may be written as :

$$\frac{d}{dt} \left\{ L - \sum_k p_k \dot{q}_k \right\} = \frac{\partial L}{\partial t}$$

or

$$\frac{d}{dt} \left\{ \sum_k p_k \dot{q}_k - L \right\} = - \frac{\partial L}{\partial t} \quad \dots(4)$$

Now ^{we} let us introduce a new function H known as *Hamiltonian function* defined as

$$H = \sum_k p_k \dot{q}_k - L(q, \dot{q}, t) \quad \dots(5)$$

It will be seen later on, that the function H is equal to the total energy in most of the physical problems. From equation (5) it is seen that Hamiltonian 'H' is the function of p , q and t , i.e.,

$$H = H(p, q, t) \quad \dots(6)$$

42.9 HAMILTON'S CANONICAL EQUATIONS OF MOTION

The Hamiltonian function H is function of p , q and t .

$$H = H(p, q, t)$$

$$= H(p_1, p_2, \dots, p_k, \dots, q_1, q_2, \dots, q_k, \dots, t)$$

$$\therefore dH = \sum_k \frac{\partial H}{\partial p_k} dp_k + \sum_k \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial t} dt \quad \dots(1)$$

Now, *Hamiltonian function*, $H = \sum p_k \dot{q}_k - L(q, \dot{q}, t)$.

$$\therefore dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - dL, \quad \dots(2)$$

Now, *Lagrangian* $L = L(q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots, t)$

$$\therefore dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

$$= \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k p_k d\dot{q}_k + \frac{\partial L}{\partial t} dt \quad \left\{ \begin{array}{l} \because \frac{\partial L}{\partial \dot{q}_k} = p_k, \\ \text{Generalised momentum} \end{array} \right. \quad \dots(3)$$

Substituting this value of dL in Eq. (2),

$$dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k p_k d\dot{q}_k - \frac{\partial L}{\partial t} dt$$

$$= \sum_k \dot{q}_k dp_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt \quad \left\{ \begin{array}{l} \because \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \quad \dots(4) \\ \text{or, } \frac{\partial L}{\partial q_k} = \frac{d}{dt} (p_k) \\ \text{or, } \frac{\partial L}{\partial q_k} = \dot{p}_k \end{array} \right.$$

Comparing coefficients of dp_k and dq_k in Eqs. (1) and (4),

$$\left. \begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned} \right\} \quad \dots(5)$$

Eqs. (5) are called *Hamilton's equations* or *Hamilton's canonical equations of motion*. p 's and q 's are called canonical variables.

Also,
$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \dots(6)$$

42.10 PHYSICAL SIGNIFICANCE OF THE HAMILTONIAN FUNCTION

The Hamiltonian, $H = H(p_1, p_2, \dots, p_k, \dots, q_1, q_2, \dots, q_k, \dots, t)$

The total time derivative of H is

$$\frac{dH}{dt} = \sum_k \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial H}{\partial p_k} \dot{p}_k + \frac{\partial H}{\partial t}$$

$$= -\sum_k \dot{p}_k \dot{q}_k + \sum_k \dot{q}_k \dot{p}_k + \frac{\partial H}{\partial t}$$

so,
$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$\therefore$ Hamilton's equations

$$\frac{\partial H}{\partial p_k} = \dot{q}_k$$

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k \quad \dots(1)$$

But
$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad \dots(2)$$

$\therefore$
$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad \dots(3)$$

If L is not an explicit function of time, $\frac{\partial L}{\partial t} = 0$. Hence

$$\frac{dH}{dt} = 0.$$

or
$$H = \text{constant.} \quad \dots(4)$$

Therefore H is a constant of the system.

For conservative systems the potential energy V does not depend upon generalised velocity, i.e.,

$$\frac{\partial V}{\partial \dot{q}_k} = 0 \quad \dots(5)$$

Also we know

$$H = \sum_k p_k \dot{q}_k - L$$

$$= \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L$$

$$= \sum_k \dot{q}_k \left\{ \frac{\partial}{\partial \dot{q}_k} (T - V) \right\} - L$$

$$= \sum_k \dot{q}_k \left\{ \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k} \right\} - L$$

$$= \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - L$$

$\therefore p_k = \frac{\partial L}{\partial \dot{q}_k}$, Generalised momentum

$\therefore \frac{\partial V}{\partial \dot{q}_k} = 0$, from Eq. (5)

$$= \sum_k \dot{q}_k \frac{\partial}{\partial \dot{q}_k} \left(\sum_k \frac{1}{2} m_k \dot{q}_k^2 \right) - L$$

$$= \sum_k m_k \dot{q}_k^2 - L$$

$$= 2T - L$$

$$= 2T - (T - V)$$

$$= T + V$$

$$= \text{K.E.} + \text{P.E.}$$

$$= E = \text{total energy of the system.}$$

{ Since, Kinetic energy

$$T = \sum_k \frac{1}{2} m_k \dot{q}_k^2$$

$$\left\{ \begin{array}{l} \therefore \frac{\partial T}{\partial \dot{q}_k} = m_k \dot{q}_k \\ \text{Only } k^{\text{th}} \text{ will differentiate} \end{array} \right.$$

Thus for conservative systems where the co-ordinate transformation is independent of time, the Hamiltonian function H represents the total energy of the system.

42.12 APPLICATIONS OF HAMILTON'S EQUATIONS OF MOTION

1. Linear Harmonic Oscillator

The system is conservative and constraint is independent of time. So Hamiltonian will represent the total energy of the system.

The kinetic energy of harmonic oscillator,

$$T = \frac{1}{2} m \dot{x}^2$$

The potential energy of harmonic oscillator,

$$\begin{aligned} V &= - \int F \, dx \\ &= \int kx \, dx = \frac{1}{2} kx^2 \end{aligned}$$

$\therefore$ The Lagrangian of the system

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = p = m\dot{x}$$

or

$$\dot{x} = p/m$$

The Hamiltonian H in terms of momenta is

$$H = T + V = \frac{1}{2} m(p/m)^2 + \frac{1}{2} kx^2 = \frac{p^2}{2m} + \frac{1}{2} kx^2$$

Equations of motion are

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx \quad \dots(1)$$

and

$$\dot{x} = \frac{\partial H}{\partial p} = p/m \quad \dots(2)$$

From Eq. (2) $\ddot{x} = \dot{p}/m = -kx/m$

$$\therefore m\ddot{x} + kx = 0 \quad \dots(3)$$

This relation shows that the motion is simple harmonic.

$$\therefore \text{The frequency } \nu = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \dots(4)$$

2. Simple Pendulum

Let the mass of the pendulum bob be m . The angle θ between rest position (OA) and deflected position (OB) is chosen as the generalized coordinate (Fig. 42.12). Let l be the length of the string.

$$\text{K.E. of the bob} = \frac{1}{2} mv^2$$

$$T = \frac{1}{2} ml^2 \dot{\theta}^2$$

P.E. of the bob is

$$\begin{aligned} V &= mgCA = mg(OA - OC) \\ &= mg(l - l \cos \theta) = mgl(1 - \cos \theta) \end{aligned}$$

The Lagrangian function L is given by

$$L = T - V = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos \theta) \quad \dots(1)$$

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \quad \dots(2)$$

The Hamiltonian is

$$\begin{aligned} H &= \sum p_j \dot{q}_j - L \\ &= p_{\theta} \dot{\theta} - L \\ &= ml^2 \dot{\theta}^2 - \left\{ \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos \theta) \right\} \\ &= \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos \theta) \\ &= T + V, \end{aligned} \quad \dots(3)$$

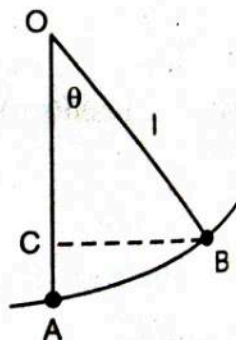


Fig. 42.12

which implies that system is conservative.

Now putting Eq. (2) into Eq. (3), we get

$$H = \frac{1}{2} ml^2 \left(\frac{p_\theta}{ml^2} \right)^2 + mgl(1 - \cos \theta),$$

$$\left[\begin{aligned} \frac{\partial H}{\partial p_\theta} &= \frac{p_\theta}{ml^2} \\ \frac{\partial H}{\partial \theta} &= mgl \sin \theta. \end{aligned} \right] \quad \dots(4)$$

Hamilton's equations of motion for θ and $\dot{p}_\theta$ are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \quad \dots(5)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl \sin \theta. \quad \dots(6)$$

Eqs. (5) and (6) represent Hamilton's equations for a simple pendulum.

From Eq. (5), we have $p_\theta = ml^2 \dot{\theta}$

or $\dot{p}_\theta = ml^2 \ddot{\theta} \quad \dots(7)$

Putting the value of p_θ from Eq. (7) into Eq. (6), we get

$$ml^2 \ddot{\theta} = -mgl \sin \theta$$

or $\ddot{\theta} + \frac{g}{l} \sin \theta = 0$

If the amplitude of motion is small, $\sin \theta \approx \theta$. Then

$$\ddot{\theta} + \frac{g}{l} \theta = 0 \quad \dots(8)$$

This gives the equation of motion of the simple pendulum.

Eq. (8) represents a S.H.M. with period

$$T = 2\pi \sqrt{l/g} \quad \dots(9)$$

3. Compound Pendulum

A rigid body capable of oscillating in a vertical plane about a fixed horizontal axis is called a compound pendulum. Let vertical plane of oscillation be xy , O the point through which axis of rotation passes, C the centre of mass. Let mass of pendulum be m , moment of inertia about axis of rotation I and distance $OC = h$ (Fig. 42.13). Let θ be the angle through which the body is displaced.

K.E. of the system is

$$T = \frac{1}{2} I \dot{\theta}^2$$

Consider the horizontal plane passing through O as reference level.

P.E. of the system is

$$V = -mg(OA) = -mgh \cos \theta$$

The Lagrangian L is written as

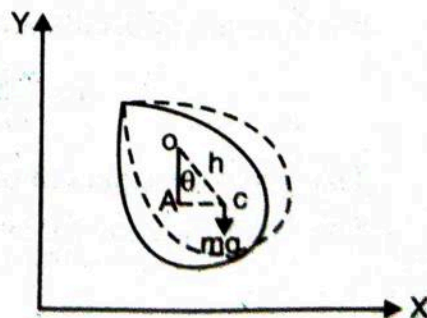


Fig. 42.13

$$L = T - V = \frac{1}{2} I \dot{\theta}^2 + mgh \cos \theta$$

Then

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta} \quad \dots(1)$$

The hamiltonian is

$$\begin{aligned} H &= \sum p_j \dot{q}_j - L = p_{\theta} \dot{\theta} - L \\ &= I \dot{\theta} \cdot \dot{\theta} - \frac{1}{2} I \dot{\theta}^2 - mgh \cos \theta \\ &= \frac{1}{2} I \dot{\theta}^2 - mgh \cos \theta \\ &= \frac{1}{2} I \left(\frac{p_{\theta}}{I} \right)^2 - mgh \cos \theta \\ &= \frac{p_{\theta}^2}{2I} - mgh \cos \theta \quad \dots(2) \\ &= T + V \end{aligned}$$

which implies that system is conservative.

Hamilton's equations of motion for θ and p_{θ} are

$$\left. \begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_{\theta}} \\ \dot{p}_{\theta} &= -\frac{\partial H}{\partial \theta} \end{aligned} \right] \quad \dots(3)$$

From Eq. (2), we find

$$\frac{\partial H}{\partial p_{\theta}} = \frac{p_{\theta}}{I}, \quad \frac{\partial H}{\partial \theta} = mgh \sin \theta.$$

Putting these in Eq. (3), we get

$$\dot{\theta} = \frac{p_{\theta}}{I} \quad \text{or} \quad \dot{p}_{\theta} = I \ddot{\theta} \quad \dots(4)$$

and

$$\dot{p}_{\theta} = -mgh \sin \theta. \quad \dots(5)$$

From Eqs. (4) and (5), we get

$$I \ddot{\theta} = -mgh \sin \theta$$

$$\text{or} \quad \ddot{\theta} + \frac{mgh}{I} \sin \theta = 0$$

If amplitude of oscillation is small, $\sin \theta \approx \theta$. Then

$$\ddot{\theta} + \frac{mgh}{I} \theta = 0 \quad \dots(6)$$

This gives the equation of motion of compound pendulum.

This is an equation for simple harmonic motion of time period

$$T = 2\pi \sqrt{\frac{I}{mgh}} \quad \dots(7)$$

Angular frequency,

$$\omega = \sqrt{mgh/I} \quad \dots(8)$$

3.6. HAMILTON'S EQUATIONS IN DIFFERENT COORDINATE SYSTEMS

(1) In cartesian coordinates : If x, y, z are the cartesian co-ordinates of a particle of mass m , then -

Kinetic energy of the particle $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$; Potential energy of the particle $V = V(x, y, z)$

$$\text{Lagrangian } L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

$$\text{Generalized momentum } p_k = \frac{\partial L}{\partial \dot{q}_k}$$

$$\text{Hence, } p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \text{ or } \dot{x} = \frac{p_x}{m}; \text{ Similarly } \dot{y} = \frac{p_y}{m} \text{ and } \dot{z} = \frac{p_z}{m}$$

$$\text{Hamiltonian } H = \sum_k p_k \dot{q}_k - L$$

For $k = x, y, z$

$$H = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z)$$

$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2} m (p_x^2 + p_y^2 + p_z^2) + V(x, y, z)$$

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \quad \dots(42)$$

Hamilton's equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

The Hamilton's equations in Cartesian coordinates are

$$\dot{x} = \frac{\partial H}{\partial p_x} \text{ or } \dot{x} = \frac{p_x}{m} \text{ or } \dot{p}_x = -\frac{\partial H}{\partial x} \text{ or } \dot{p}_x = -\frac{\partial V}{\partial x} \quad \dots(43a)$$

$$\text{Similarly, } \dot{y} = \frac{p_y}{m} \text{ and } \dot{p}_y = -\frac{\partial V}{\partial y} \quad \dots(43b)$$

$$\dot{z} = \frac{p_z}{m} \text{ and } \dot{p}_z = -\frac{\partial V}{\partial z} \quad \dots(43c)$$

Thus one may express equations of motion as

$$m\ddot{x} = -\frac{\partial V}{\partial x}, \quad m\ddot{y} = -\frac{\partial V}{\partial y}, \quad m\ddot{z} = -\frac{\partial V}{\partial z} \quad \dots(44)$$

(2) In polar coordinates : If r, θ are the polar coordinates of a particle of mass m , then

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

Hence,

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta \quad \text{and} \quad \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta$$

Now,

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \text{ and } L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r, \theta)$$

Generalized momenta $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta}$ or $\dot{r} = \frac{p_r}{m}$ and $\dot{\theta} = \frac{p_\theta}{mr^2}$

Thus,

$$H = \sum_k p_k \dot{q}_k - L = p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r, \theta)$$

or

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r, \theta) \quad \dots(45)$$

Hamilton's equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \& \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} \text{ or } \dot{r} = \frac{p_r}{m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} \text{ or } \dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \quad \dots(46a)$$

and

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ or } \dot{\theta} = \frac{p_\theta}{mr^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} \text{ or } \dot{p}_\theta = -\frac{\partial V}{\partial \theta} \quad \dots(46b)$$

(3) In cylindrical coordinates : Here, $x = r \cos \theta, y = r \sin \theta, z = z$

Hence

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) - V(r, \theta, z)$$

Generalized momentum $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ or $\dot{r} = \frac{p_r}{m}$; Similarly, $\dot{\theta} = \frac{p_\theta}{mr^2}, \dot{z} = \frac{p_z}{m}$

Now,

$$H = \sum_k p_k \dot{q}_k - L, \text{ where } k = r, \theta, z$$

Thus,

$$H = p_r \dot{r} + p_\theta \dot{\theta} + p_z \dot{z} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + V(r, \theta, z)$$

$$= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2 \right) + V(r, \theta, z) \quad \dots(47)$$

Hamilton's equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

Therefore,

$$\dot{r} = \frac{\partial H}{\partial p_r} \text{ or } \dot{r} = \frac{p_r}{m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} \text{ or } \dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \quad \dots(48a)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_{\theta}} \text{ or } \dot{\theta} = \frac{p_{\theta}}{m} \text{ and } \dot{p}_{\theta} = -\frac{\partial H}{\partial \theta} \text{ or } \dot{p}_{\theta} = -\frac{\partial V}{\partial \theta} \quad \dots(48b)$$

$$\dot{z} = \frac{\partial H}{\partial p_z} \text{ or } \dot{z} = \frac{p_z}{m} \text{ and } \dot{p}_z = -\frac{\partial H}{\partial z} \text{ or } \dot{p}_z = -\frac{\partial V}{\partial z} \quad \dots(48c)$$

(4) In spherical coordinates : Here

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

Hence,

$$\dot{x} = \dot{r} \sin \theta \cos \phi + r \dot{\theta} \cos \theta \cos \phi - r \sin \theta \dot{\phi} \sin \phi$$

$$\dot{y} = \dot{r} \sin \theta \sin \phi + r \dot{\theta} \cos \theta \sin \phi + r \sin \theta \dot{\phi} \cos \phi$$

$$\dot{z} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta$$

$$\text{Thus, } L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \text{ or } L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V(r, \theta, \phi)$$

$$\text{Therefore, } p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ or } \dot{r} = \frac{p_r}{m}; \text{ Similarly } \dot{\theta} = \frac{p_{\theta}}{mr^2} \text{ and } \dot{\phi} = \frac{p_{\phi}}{mr^2 \sin \theta}$$

$$\text{Now } H = \sum_{k=r, \theta, \phi} p_k \dot{q}_k - L = p_r \dot{r} + p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V(r, \theta, \phi)$$

$$\text{or } H = \frac{1}{2m} \left(p_r^2 + \frac{p_{\theta}^2}{r^2} + \frac{p_{\phi}^2}{r^2 \sin^2 \theta} \right) + V(r, \theta, \phi) \quad \dots(49)$$

Hamiltonian equations $\left(\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k} \right)$ are

$$\dot{r} = \frac{\partial H}{\partial p_r} \text{ or } \dot{r} = \frac{p_r}{m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} \text{ or } \dot{p}_r = -\frac{p_{\theta}^2}{mr^3} - \frac{p_{\phi}^2}{mr^3 \sin^2 \theta} - \frac{\partial V}{\partial r} \quad \dots(50a)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_{\theta}} \text{ or } \dot{\theta} = \frac{p_{\theta}}{mr^2} \text{ and } \dot{p}_{\theta} = -\frac{\partial H}{\partial \theta} \text{ or } \dot{p}_{\theta} = \frac{p_{\phi}^2 \cos \theta}{mr^2 \sin^3 \theta} - \frac{\partial V}{\partial \theta} \quad \dots(50b)$$

$$\dot{\phi} = \frac{\partial H}{\partial p_{\phi}} \text{ or } \dot{\phi} = \frac{p_{\phi}}{mr^2 \sin^2 \theta} \text{ and } \dot{p}_{\phi} = -\frac{\partial H}{\partial \phi} \text{ or } \dot{p}_{\phi} = -\frac{\partial V}{\partial \phi} \quad \dots(50c)$$