

Ques ① Define Automorphism of group. ①

Sol - Let  $G$  be a group. Then a mapping  $f: G \rightarrow G$  is said to be an automorphism if it is one-one onto and  $f(ab) = f(a)f(b) \quad \forall a, b \in G$ .

i.e. An isomorphism  $f: G \rightarrow G$  is known as Automorphism.

Ques ② Define conjugacy class in a group.

Sol - Let  $G$  be a group and  $a, b \in G$ . Then  $b$  is said to be a conjugate of  $a$  in  $G$  if there exist an element  $x \in G$  such that  $b = x^{-1}ax$ . It's denoted as  $a \sim b$  and this relation is known as relation of conjugacy.

Ques ③ Define Euclidean domain.

Sol - Let  $R$  be an integral domain. Then this integral domain  $R$  is said to be a Euclidean domain if for every non-zero element  $a \in R$  there exist a non-negative integer  $d(a)$  such that

(i) for all non-zero elements  $a, b \in R$ ,  $d(a) \leq d(a, b)$  or  $d(b) \leq d(a, b)$

(ii) for each  $a, b \in R$  with  $b \neq 0$ , there exist  $q, r \in R$  such that  $a = bq + r$  where either  $r = 0$  or  $d(r) < d(b)$ .

Ques ④ Define linear functional.

Sol - Let  $V$  be a vector space over a field  $F$ .

A linear functional on  $V$  is a linear transformation  $f$  from  $V$  into the field of scalars  $F$ .

i.e. a mapping  $f: V \rightarrow F$  such that for all  $u, v \in V$  and  $a, b \in F$

$$f(au + bv) = af(u) + bf(v)$$

Ex  $V = \mathbb{R}^2$ ,  $f(x, y) = 2x + 3y \quad f: V \rightarrow \mathbb{R}$

Ques 5 If  $v_1$  and  $v_2$  are vectors in  $V$  and  $a, b \in F$  then show that the set  $\{v_1, v_2, av_1 + bv_2\}$  is linearly dependent.

Sol - set of vectors  $\{v_1, v_2, av_1 + bv_2\} \exists$  scalars  $c_1, c_2, c_3$  not all zero  $\Rightarrow$  L.D.  $c_1 v_1 + c_2 v_2 + c_3 (av_1 + bv_2)$  Here take  $c_1 = a, c_2 = b, c_3 = -1$   
we find non-zero scalars  $\Rightarrow$  L.D.

Ques 6 Define basis of a Vector space

Sol - Let  $V$  be a vector space over a field  $F$ . A subset  $B$  of  $V$  is called a basis of  $V$  if (i)  $B$  is L.I.

(ii)  $B$  spans  $V$  i.e.  $L(B) = V$ . Ex  $\rightarrow V = \mathbb{R}^2$ , set  $\{(1,0), (0,1)\}$  basis

Ques 7 write the quadratic form of the matrix  $\begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$

Sol - For a symmetric matrix  $A = [a_{ij}]$ ,  $Q = \sum a_{ii} x_i^2 + 2 \sum_{i < j} a_{ij} x_i x_j$   
 $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$   $Q = 2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$ .

Ques 8 State Cauchy-Schwarz inequality.

Sol - Let  $V$  be an inner product space. Then for any vectors  $u, v \in V$   $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$  where  $\|u\| = \sqrt{\langle u, u \rangle}$ .

Ques 9 Define principal ideal domain with an example?

Sol - An ideal  $I$  of a ring  $R$  is called a principal ideal if it is generated by a single element. i.e.  $I = \langle a \rangle = \{ra : r \in R\}$  for some  $a \in R$ .

Ex - Any field  $F$  is a PID.

Ques 10 Find the no. of 7-sylow subgroups of the group of order 168.

Sol -  $|G| = 168 = 2^3 \cdot 3 \cdot 7$ . let  $n_7$  be no. of 7-sylow subgroups

(i)  $n_7 \mid \frac{|G|}{7} = 24$  (ii)  $n_7 \equiv 1 \pmod{7}$

Divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24

(2)

$$1 \equiv 1 \pmod{7} \quad 0 \equiv 1 \pmod{7} \quad \text{so } n_7 = 1 \text{ or } 8$$

if  $n_7 = 1 \Rightarrow$  7-Sylow subgroup is unique hence Normal.

$n_7 = 8 \Rightarrow$   $\exists$  eight 7-Sylow subgroups.

Ques (11) Let  $R^+$  be the multiplicative group of all +ve real numbers. Show that the mapping  $f: R^+ \rightarrow R^+$  defined by  $f(x) = x^2 \forall x \in R^+$  is an automorphism.

Sol- Homomorphism  $\rightarrow x, y \in R^+ \Rightarrow f(xy) = (xy)^2 = x^2 y^2 = f(x)f(y) \Rightarrow f$  is homomorphism

For one-one

$$\text{Take } f(x) = f(y) \text{ for } x, y \in R^+ \Rightarrow x^2 = y^2 \Rightarrow x^2 y^2 = 0$$

$$\Rightarrow (x-y)(x+y) = 0 \text{ since } x, y > 0, x+y \neq 0 \text{ so } x-y = 0 \Rightarrow x = y$$

For onto  $y \in R^+$  we need to find  $x \in R^+$  such that  $f(x) = y$

Take  $x = \sqrt{y}$  since  $y > 0, \sqrt{y} > 0$  and  $\sqrt{y} \in R^+$  Then  $f(x) = (\sqrt{y})^2 = y$ . So for every  $y \exists$  a pre-image hence onto  $\Rightarrow$  automorphism.

Ques (12) Prove that the intersection of two ideals of a ring  $R$  is an ideal of  $R$ .

Sol- Let  $I$  and  $J$  be two ideals of Ring  $R$ . Let  $K = I \cap J$

we have to show for  $K$  is again a ideal (i)  $K$  is subgroup under +.

(ii) For  $r \in R$  and  $a \in K, ra \in K$  and  $ar \in K$  <sup>For Proof</sup> Take  $a, b \in K$

Now  $a, b \in I, a, b \in J$  both are ideal so  $a-b \in I, a-b \in J \Rightarrow a-b \in K \Rightarrow$  Subgroup

For (ii) Let  $x \in R, a \in K, ra \in I, ar \in I, ra \in J, ar \in J \Rightarrow ar, ra \in I \cap J = K$ .

$$\Rightarrow a, x \in I \cap J = K \Rightarrow I \cap J \text{ is an ideal of } R.$$

Ques (13) Show that the set  $W = \{ (x, y, 0) ; x, y \in F \}$  is a subspace of  $V_3(F)$ .

Sol- Let  $V_3(F) = \{ (x_1, x_2, x_3) : x_i \in F \}$  be the vector space over field  $F$ .  $W = \{ (x, y, 0) : x, y \in F \}$ .

A non-empty subset  $W$  of  $V$  is a subspace iff  $c\alpha + \beta \in W$  for all  $\alpha, \beta \in W$  and  $c \in F$ .  $0 \in W \Rightarrow W$  is non empty.

$$c\alpha + \beta = c(x_1, y_1, 0) + (x_2, y_2, 0) = (cx_1, cy_1, 0) + (x_2, y_2, 0)$$

$$= (cx_1 + x_2, cy_1 + y_2, 0) \text{ since } F \text{ is a field}$$

$cx_1 + x_2 \in F$  and  $cy_1 + y_2 \in F$  Hence  $c\alpha + \beta \in W \Rightarrow W$  is subspace of  $V_3(F)$

Ques (14) Define rank and nullity of a linear transformation

Sol - Let  $U$  and  $V$  be vector spaces over a field  $F$  and let  $T: U \rightarrow V$  be a linear transformation

$$\text{Range} \rightarrow R(T) = \{T(u) : u \in U\} = \text{Im}(T)$$

Rank - The dimension of the range of  $T$  is called the rank of  $T$ .

$$\text{rank}(T) = \dim(R(T))$$

Nullity - The dimension of the Kernel of  $T$  is called the nullity of  $T$ .

$$\text{nullity}(T) = \dim(N(T)) = \dim(\ker(T))$$

Ques (15) Show that the three vectors  $(1, 1, -1)$ ,  $(2, -3, 5)$  and  $(-2, 1, 4)$  of  $R^3$  are linearly independent.

Sol -  $\alpha(1, 1, -1) + \beta(2, -3, 5) + \gamma(-2, 1, 4) = (0, 0, 0)$

$$\alpha + 2\beta - 2\gamma = 0 \Rightarrow \alpha + 2\beta - 2\gamma = 0 \quad \text{--- (i)} \quad 2\beta + 5\gamma = 0 \quad \text{--- (v)}$$

$$\alpha - 3\beta + \gamma = 0 \quad \text{--- (ii)} \quad \text{Solving (iv) + (v) eq. wgt}$$

$$-\alpha + 5\beta + 4\gamma = 0 \quad \text{--- (iii)} \quad 5\beta - 3\gamma = 0 \quad \text{--- (iv)} \quad \alpha = \beta + \gamma = 0 \Rightarrow \text{L.I.}$$

Ques (16) Define inner automorphism.

Sol - Let  $a$  be any fixed element of a group  $G$ . Define a mapping  $f_a: G \rightarrow G$  by  $f_a(x) = a^{-1}xa \quad \forall x \in G$ , then for any  $x, y \in G$

(i)  $f_a(xy) = f_a(x)f_a(y)$  (ii)  $f_a$  is injective (iii)  $f_a$  is surjective.

Define associates element.

Ques (17) Let  $R$  be an integral domain with unit elements. Two non-zero elements  $a, b \in R$  are associates iff  $a/b \mid b/a$ .

Ques (18) Define inner product space.

Sol - Let  $V$  be a vector space over a field  $F$

An inner product on  $V$  is a function which assigns to each pair of vectors  $u, v \in V$  a scalar  $\langle u, v \rangle \in F$  satisfying the following axioms; (i) conjugate symmetry  $\langle u, v \rangle = \overline{\langle v, u \rangle}$

$$(ii) \langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle \quad (iii) \text{ +ve definite}$$

Linearity

$$\langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \Leftrightarrow u = 0$$

Ques 19 Define Unique factorization domain. ③

Sol - An integral domain  $R$  is called a unique factorization domain if (i) every non-zero element of  $R$  is either a unit or can be expressed as a product of a finite no. of irreducible elements of  $R$ .

(ii) this irreducible factorization is unique upto the associates and their order.

Ques 20 If  $\{\alpha_1, \alpha_2, \alpha_3\}$  is a basis of a vector space  $V_3(K)$ . Show that  $\{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_1\}$  is also a basis of  $V_3(K)$ .

Sol - Let  $\beta_1 = \alpha_1 + \alpha_2$ ,  $\beta_2 = \alpha_2 + \alpha_3$ ,  $\beta_3 = \alpha_3 + \alpha_1$ , and  $B' = \{\beta_1, \beta_2, \beta_3\}$  we have to show that  $B'$  is a basis. It's enough to show  $B'$  is L.I because no. of vectors  $= 3 = \dim V_3$ .

$$\text{So } a\beta_1 + b\beta_2 + c\beta_3 = 0 \Rightarrow (a+c)\alpha_1 + (a+b)\alpha_2 + (b+c)\alpha_3 = 0$$

Since  $\{\alpha_1, \alpha_2, \alpha_3\}$  is L.I  $\Rightarrow$  coefficient must be zero  $\begin{matrix} a+c=0 \\ a+b=0 \\ b+c=0 \end{matrix}$   
 $\Rightarrow$  Solving these eq. we get  $a=b=c=0 \Rightarrow B'$  is L.I set.

Ques 21 Define Sylow's  $p$ -subgroup with an example.

Sol - Def - Let  $G$  be a finite group of order  $p^m n$  where  $m$  and  $n$  are +ve integers and  $p$  is a prime no. such that  $p$  does not divide  $n$ . Then a subgroup  $H$  of order  $p^m$  is called a Sylow  $p$ -subgroup of  $G$ .

Ex  $\rightarrow O(G) = 48$  has a Sylow 2-subgroup.

Ques 22 Prove that every field is principal ideal ring.

Sol <sup>case 1</sup> - If  $S$  is null ideal i.e.  $S = (0)$  then we have  $0 \cdot f(x) = 0 \forall f(x) \in F[x]$   
In this case  $S$  is an ideal generated by  $0 \Rightarrow$  P.I.

case 2 If  $S \neq 0$  then  $\exists$  a non-zero polynomial  $g(x)$  of lowest degree in  $S$   
we claim  $S = (g(x))$  means  $S$  generated by  $g(x)$ .

$f(x) \in S$  then by division algorithm  $\exists$  two unique polynomial  $q(x)$  &  $r(x) \Rightarrow f(x) = q(x)g(x) + r(x)$  where  $r(x) = 0$  or  $\deg r(x) < \deg g(x)$

Since  $S$  is an ideal of  $F[x]$  then  $g(x) \in S$ ,  $q(x) \in F[x] \Rightarrow q(x)g(x) \in S$   
Again  $f(x) \in S$ ,  $q(x)g(x) \in S \Rightarrow \{f(x) - q(x)g(x)\} \in S \Rightarrow r(x) \in S$ .

which contradict because  $g(x)$  is the lowest degree of polynomial  
therefore  $r(x) = 0$  so  $S$  generated by  $g(x)$  Hence Principal ideal.

Ques 23 Define Euclidean domain and give an example.

Sol - Let  $R$  be an integral domain then  $R$  is said to be Euclidean domain if for every non-zero element  $a \in R$   $\exists$  non-negative integer  $d(a)$  such that (i) for all non-zero elements  $a, b \in R$ ,  $d(a) \leq d(a, b)$  or  $d(b) \leq d(a, b)$

(ii) for each  $a, b \in R$  with  $b \neq 0$   $\exists q, r \in R$  such that  $a = bq + r$  where either  $r = 0$  or  $d(r) < d(b)$ .

Ques 24 Show that  $a \rightarrow a^{-1}$  is an automorphism of a group  $G$  iff  $G$  is an abelian.

Sol - suppose  $\phi$  is an automorphism. Conversely  $G$  is abelian to show that  $\phi$  is an automorphism.

$x, y \in G \Rightarrow xy \in G \Rightarrow \phi(xy) = (xy)^{-1} = y^{-1}x^{-1}$   
 $\phi(x)\phi(y) = y^{-1}x^{-1}$  ( $\because \phi$  is Homomorphism)  $\Rightarrow \phi$  is homomorphism.

$x^{-1}y^{-1} = y^{-1}x^{-1} \Rightarrow (y^{-1})^{-1}(x^{-1})^{-1} = (x^{-1})^{-1}(y^{-1})^{-1}$   
 $\Rightarrow y(x) = x(y) \Rightarrow G$  is abelian. Again  $\phi(x) = \phi(y) \Rightarrow x^{-1} = y^{-1} \Rightarrow (x^{-1})^{-1} = (y^{-1})^{-1}$   
 $\Rightarrow x = y \Rightarrow \phi$  is one to one.

For onto  $x, x^{-1} \in G \Rightarrow x = (x^{-1})^{-1} = \phi(x^{-1})$   
 for  $x \in G \exists x^{-1} \in G$  such that  $x = \phi(x^{-1})$ .  
 $\Rightarrow$  thus  $\phi$  is onto.

Ques 25 Let  $W = \{a \in \mathbb{R}^3 : a = (a_1, a_2, a_3)\}$  such that  $a_1 > 0$ .

Determine whether  $W$  is a subspace of  $\mathbb{R}^3$  or not.

Sol -  $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 > 0\}$ . For a subspace two conditions must hold (i)  $0 \in W$  (ii)  $c\alpha + \beta \in W \forall \alpha, \beta \in W, c \in \mathbb{R}$ .

Here  $(0, 0, 0) \in W$  so 1<sup>st</sup> condition holds good.

But second not holds because

$\alpha = (1, 0, 0) \in W$  Take scalar  $c = -1 \in \mathbb{R}$ . Then  $c\alpha = -1(1, 0, 0) = (-1, 0, 0)$

First co-ordinate  $-1 \not> 0$  so  $c\alpha \notin W \Rightarrow W$  is not closed under scalar multiplication  $\Rightarrow W$  is not a subspace.

Ques 26 Show that the mapping  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $f(x, y) = |x - y|$  is not linear.

Sol. A mapping  $f: V \rightarrow V$  is linear if

(1)  $f(\alpha + \beta) = f(\alpha) + f(\beta)$  (2)  $f(c\alpha) = cf(\alpha) \forall \alpha, \beta \in V, c \in \mathbb{R}$

Check condition (2) Take  $\alpha = (1, 0) \in \mathbb{R}^2$  and  $c = -1$

then  $c\alpha = (-1, 0) \Rightarrow f(c\alpha) = f(-1, 0) = |-1 + 0| = |-1| = 1$

$cf(\alpha) = -1 \cdot f(1, 0) = -1 \cdot |1 + 0| = -1 \cdot 1 = -1$  so  $f(c\alpha) \neq cf(\alpha)$

similarly  $f(\alpha + \beta) = 0$  and  $f(\alpha) + f(\beta) = 2 \Rightarrow f$  is not linear.

Ques (17) Let  $V(K)$  be a finite dimensional vector space (4).  
Show that  $\dim V^* = \dim V$  where  $V^*$  denote dual space of  $V$ .

Sol - Let  $V$  be  $n$ -dimensional vector space. Let  $V^*$  be its dual space, so that

$$\dim V = n, \quad L(V, F) = V^* \quad \dim F = 1.$$

Since we know that

$$\dim L(U, V) = (\dim U) (\dim V)$$

$$\dim L(V, F) = (\dim V) (\dim F)$$

$$\dim V^* = (\dim V) \cdot 1 \Rightarrow \dim V^* = \dim V.$$

Ques (20) Define Normed Vector space.

Sol - Let  $V$  be a vector space over a field  $K$ . A norm on  $V$  is a real valued fun.  $x \rightarrow \|x\|$  defined on  $V$

satisfying (i)  $\|x\| \geq 0 \quad \forall x \in V$  and  $\|x\| = 0 \Leftrightarrow x = 0$

(ii)  $\|\alpha x\| = |\alpha| \|x\| \quad \forall x \in V, \alpha \in K.$

(iii)  $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V.$

Then  $V$  equipped with this norm is called a Normed Vector space, denoted by  $(V, \|\cdot\|)$ .

Ques (29) If  $P$  is the only Sylow  $p$ -subgroup of  $G$ , then  $P$  is normal in  $G$  and conversely.

Sol - Let  $|G| = p^m q$  where  $p \nmid q$  and  $P$  be a  $p$ -Sylow subgroup so  $|P| = p^m$ . We know that all Sylow  $p$ -subgroups are conjugate to each other i.e.  $Q = gPg^{-1}$  for some  $g \in G$ .

$P$  is only Sylow-subgroup  $\forall g \in G \Rightarrow |gPg^{-1}| = |P| = p^m$  and  $gPg^{-1}$  is a subgroup of  $G \Rightarrow gPg^{-1} = P$  for all  $g \in G \Rightarrow$  Normal subgroup.

Converse Assume  $P \triangleleft G$  i.e.  $gPg^{-1} = P$  for all  $g \in G$ .  $Q$  be any other Sylow  $p$ -subgroup  $\Rightarrow Q = gPg^{-1}$  for some  $g \in G$ . But  $gPg^{-1} = P$  by normality  $\Rightarrow Q = P$  so  $\nexists$  any other Sylow  $p$ -subgroup  $\Rightarrow P$  is unique.

Ques 30 Determine whether fun.  $f$  defined on vector's  $\alpha = (x_1, x_2)$  and  $\beta = (y_1, y_2)$  in  $\mathbb{R}^2$  such that  $f(\alpha, \beta) = (x_1 - y_1)^2 + x_1 y_2$  is a bilinear map or not.

Sol - A map  $f: V \times V \rightarrow F$  is called bilinear if it is linear in each argument separately i.e.

$$(i) f(c\alpha_1 + \alpha_2, \beta) = c f(\alpha_1, \beta) + f(\alpha_2, \beta).$$

$$(ii) f(\alpha, c\beta_1 + \beta_2) = c f(\alpha, \beta_1) + f(\alpha, \beta_2); \alpha_1, \alpha_2, \alpha, \beta, \beta_1, \beta_2 \in V, c \in F.$$

$$\alpha_1 = (x_1, x_2) \quad \alpha_2 = (x_1', x_2'), \quad \beta = (y_1, y_2).$$

$$f(\alpha_1 + \alpha_2, \beta) = ((x_1 + x_1' - y_1)^2 + (x_2 + x_2')y_2) \\ = (x_1 + x_1' - y_1)^2 + x_2 y_2 + x_2' y_2 \quad \text{both are not equal.}$$

Ques 31 Define linearly independent set with example.

Sol - A set  $S$  is said to be linearly independent if

Let  $V(F)$  be a vector space. A finite set of vectors  $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$  in  $V$  is said to be L.I. if  $c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n = 0 \Rightarrow c_1 = c_2 = \dots = 0$ : all are zero.

Ques 32 Show that  $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$  is a basis of  $\mathbb{R}^3(\mathbb{R})$ .

Sol -  $S = \{\alpha_1 = (1, 1, 1), \alpha_2 = (1, 1, 0), \alpha_3 = (1, 0, 0)\}$ .

$\dim \mathbb{R}^3 = 3$  so  $S$  has three linearly independent vectors  $\alpha(1, 1, 1) + \beta(1, 1, 0) + \gamma(1, 0, 0) = (0, 0, 0)$ .

$$(\alpha + \beta + \gamma, \alpha + \beta, \alpha) = (0, 0, 0) \text{ solve these eq.}$$

$$\alpha + \beta + \gamma = 0$$

$$\alpha + \beta = 0$$

$$\boxed{\alpha = 0} \Rightarrow \boxed{\beta = 0} = \gamma = 0 \Rightarrow \text{L.I. Now we have to show } S \text{ spans } \mathbb{R}^3.$$

$$(\alpha + \beta + \gamma, \alpha + \beta, \alpha) = (x, y, z).$$

$$\Rightarrow \alpha + \beta = y, \alpha = z. \Rightarrow \boxed{\beta = y - z}$$

$$\alpha + \beta + \gamma = x \Rightarrow \gamma = x - z - y + z = x - y$$

Hence  $S$  spans  $\mathbb{R}^3$ , Since  $S$  is L.I and spans  $\mathbb{R}^3$ ,  $S$  is a basis of  $\mathbb{R}^3$ .

Ques 33 Show that  $2x+1$  is a unit in  $\mathbb{Z}_4[x]$ . (5)

Sol - since  $\mathbb{Z}_4 = \{0, 1, 2, 3, 4\}$  then  $2x+1 \in \mathbb{Z}_4[x]$

$$\text{Now } (2x+1)(2x+1) = 4x^2 + 4x + 1 = 0x^2 + 0x + 1 = 1 \in \mathbb{Z}_4$$

As 1 is the unity in  $\mathbb{Z}_4$  so that  $2x+1$  has multiplicative inverse in  $\mathbb{Z}_4[x]$ , hence  $2x+1$  is a unit in  $\mathbb{Z}_4[x]$ .

Ques 34 Prove that  $x^2+x+4$  is irreducible over  $F$ , the field of integers mod 11 and prove directly that  $F[x]/x^2+x+4$  is a field having 121 elements.

Sol - since  $F = \{0, 1, 2, \dots, 10\} \pmod{11}$  then  $x^2+x+4 \in F[x]$ .

$\nexists \alpha \in F$  such that  $\alpha^2+\alpha+4=0$  therefore  $x^2+x+4$  is irreducible over  $F$ , hence  $F[x]/x^2+x+4$  is a field.

$\deg(x^2+x+4) = 2$  and  $p=11$  thus  $F[x]/x^2+x+4$  has  $p^n = (11)^2 = 121$

Ques 35  $R(C)$  is not a vector space Give the reason.

Sol - Take  $V=R$  and  $F=C$

Let  $v=1 \in R$  and  $c=i \in C$  then  $cv = i \cdot 1 = i \notin R$

Scalar multiplication is not closed in  $R \Rightarrow R$  is not a vector space over  $C$ .

Ques 36 Prove that center of group  $G$  is a subgroup of  $G$ .

Sol - center of  $G$  is  $Z(G) = \{z \in G, zg = gz \ \forall g \in G\}$ .

To prove  $Z(G) \leq G$ . (i)  $Z(G) \neq \emptyset \Rightarrow eg = g = ge \Rightarrow e \in Z(G)$

Closure -  $(z_1 z_2)g = z_1(z_2 g) = z_1(g z_2) = (z_1 g) z_2 = (g z_1) z_2 \Rightarrow z_1 z_2 \in Z(G)$

Inverse  $(zg)^{-1} = (gz)^{-1} \Rightarrow g^{-1}z^{-1} = z^{-1}g^{-1} \ \forall g$  so  $z^{-1} \in Z(G)$ .

$\Rightarrow$  Thus  $Z(G)$  satisfies subgroup criteria, so  $Z(G)$  is a subgroup.

Ques 37 Define irreducible and prime element in a ring.

Sol - An element  $a \in R$  is called irreducible if whenever  $a = bc$  for  $b, c \in R$  then either  $b$  is a unit or  $c$  is a unit.

Prime element - An element  $p \in R$  is called prime if  $p|ab \Rightarrow p|a$  or  $p|b$  for any  $a, b \in R$

Que 38 Define algebraic and geometric multiplicity of a characteristic root.

Sol - Let  $T: V \rightarrow V$  be a linear operator on finite dimensional vector space and  $c$  be a characteristic root.

$f(x) = |T - xI| \rightarrow$  ch. Polynomial  $(x-c)^k$  is the highest power of  $(x-c)$  then  $f(x) = (x-c)^k g(x)$ ,  $g(c) \neq 0$   $k$  is algebraic multiplicity.

Que 39 dim of eigen space of  $c$   $W_c = \{v \in V : T(v) = cv\} = \ker(T - cI)$  is called geometric multiplicity of  $c$ .  
 $1 \leq G.M \leq A.M$  for any eigenvalue.

Que 39 The set  $W$  of ordered triads  $(\alpha_1, \alpha_2, 0)$  where  $\alpha_1, \alpha_2 \in F$  is a subspace of  $V_3(F)$ .

Sol -  $V_3(F) = \{(x_1, x_2, x_3) : x_i \in F\}$ ,  $W = \{(a_1, a_2, 0) : a_i \in F\}$   
 we need to show that  $W$  is non-empty and closed under addition and scalar multiplication.

(i)  $0 = (0, 0, 0) \in W$  so  $W \neq \emptyset$

(ii) Let  $\alpha = (\alpha_1, \alpha_2, 0) \in W$  and  $\beta = (\beta_1, \beta_2, 0) \in W, c \in F$ .

Then  $c\alpha + \beta = c(\alpha_1, \alpha_2, 0) + (\beta_1, \beta_2, 0) = (c\alpha_1 + \beta_1, c\alpha_2 + \beta_2, 0)$

so  $W$  is closed under linear combination.

Hence  $W$  is a subspace of  $V_3(F)$ .

Que 40 If  $\alpha, \beta$  be vectors in real inner product space such that  $\|\alpha\| = \|\beta\|$  then prove that

$$(\alpha + \beta, \alpha - \beta) = 0$$

Sol - Let  $V$  be a inner product space satisfying (i)  $\langle x, y \rangle = \langle y, x \rangle$   
 we have to show that  $\alpha + \beta$  is  $\perp$  to  $\alpha - \beta$ . (ii)  $\langle x, x \rangle = \|x\|^2 \geq 0$

$$(\alpha + \beta, \alpha - \beta) = (\alpha, \alpha - \beta) + (\beta, \alpha - \beta) = (\alpha, \alpha) - (\alpha, \beta) + (\beta, \alpha) - (\beta, \beta)$$

In real inner product space  $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$ .

$$\text{So } (\alpha + \beta, \alpha - \beta) = (\alpha, \alpha) - (\beta, \beta) = \|\alpha\|^2 - \|\beta\|^2$$

$$\text{Given } \|\alpha\| = \|\beta\| \Rightarrow \|\alpha\|^2 = \|\beta\|^2 \Rightarrow \langle \alpha + \beta, \alpha - \beta \rangle = 0.$$

$\Rightarrow \alpha + \beta$  is orthogonal to  $\alpha - \beta$ .

Ques 41 Let  $T$  be a linear transformation from a v.s  $U(F)$  into a v.s  $V(F)$  then  $T$  is non-singular iff  $T$  is one-one.

Sol -  $T: U \rightarrow V$  is non-singular if  $\ker(T) = \{0\}$  i.e.  $T(u) = 0_V$   
 $T$  is one-one  $\Rightarrow T(\alpha_1) = T(\alpha_2) \Rightarrow \alpha_1 = \alpha_2 \forall \alpha_1, \alpha_2 \in U$ .

First  $T$  is non-singular  $\Rightarrow T$  is one-one.

$$T(\alpha_1) = T(\alpha_2) \Rightarrow T(\alpha_1) - T(\alpha_2) = 0 \Rightarrow T(\alpha_1 - \alpha_2) = 0 \quad (\because T \text{ is linear})$$

Since  $T$  is non-singular, only zero vector maps to zero  
 So  $\alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2 \Rightarrow$  one-one.

Converse  $T$  is one-one  $\Rightarrow T$  is non-singular

$$\alpha \in U : T(\alpha) = 0_V \quad \text{But } T(0_U) = 0_V \text{ also so } T(\alpha) = T(0_U)$$

Since  $T$  is one-one  $\alpha = 0_U$  Thus  $\ker(T) = \{0_U\}$

Hence  $T$  is non-singular.

Ques 42 Add and multiply the following polynomials over the ring  $(I_5, +, \cdot)$ .

Sol -  $f(x) = 3 + 4x + 2x^2$      $g(x) = 1 + 3x + 4x^2 + 2x^3$

$$f(x) + g(x) = 2x^3 + 6x^2 + 7x + 4 = 2x^3 + x^2 + 2x + 4.$$

$$f(x) \cdot g(x) = 3 + 8x + 6x^2 + 3x^3 + 6x^4 + 4x^5 \pmod{5}$$

$$f(x) \cdot g(x) = 3 + 3x + x^2 + 3x^3 + x^4 + 4x^5.$$

Ques 43 Define polynomial ring over commutative ring.

Sol - Let  $R$  be a commutative ring with unity. Then the set

$$R[x] = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n : a_i \in R\}.$$

It's called a polynomial ring over  $R$ .

$(R[x], +, \cdot)$  forms a commutative ring with unity containing  $R$  as a subring.

Ques 44 Show that the mapping  $T$  defined by  
 $T(\alpha, \beta) = (\alpha + \beta, \alpha - \beta, \beta) \forall (\alpha, \beta) \in V_2(R)$  is a L.T.

Sol -  $T: V_2(R) \rightarrow V_3(R)$   
 For each  $a, b \in F$  and  $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in V_2(R)$ .  
 $T[a(\alpha_1, \beta_1) + b(\alpha_2, \beta_2)] = T(a\alpha_1 + b\alpha_2, a\beta_1 + b\beta_2)$   
 $= a(\alpha_1 + \beta_1, \alpha_1 - \beta_1, \beta_1) + b(\alpha_2 + \beta_2, \alpha_2 - \beta_2, \beta_2)$   
 $= aT(\alpha_1, \beta_1) + bT(\alpha_2, \beta_2) \Rightarrow T \text{ is linear.}$

Ques 45 Show that the set  $S = \{(1, 2), (3, 4)\}$  forms a basis of  $R^2$ .

Sol  $\rightarrow \dim R^2 = 2$  and  $S$  contains 2 elements. So we have to show that  $S$  is L.I.

$$a(1, 2) + b(3, 4) = (0, 0) \Rightarrow a + 3b = 0, 2a + 4b = 0.$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \Rightarrow |A| = 4 - 6 = -2 \neq 0.$$

Rank  $A = 2$  which is equal to no. of unknowns.

$\Rightarrow \exists$  only zero solution  $a=0, b=0 \Rightarrow S$  is L.I.  
 $\Rightarrow S$  forms a basis of  $R^2$ .

Ques 46 Prove that  $\langle x \rangle$  is a maximal ideal of  $Q[x]$  where  $Q$  is the field of all rational no.

Sol - we know that if  $R$  commutative ring then  
 $\frac{R[x]}{\langle x \rangle} \cong R$ . Since  $Q$  is a field and  $Q \cong \frac{Q[x]}{\langle x \rangle}$

So that  $\frac{Q[x]}{\langle x \rangle}$  is a field hence  $\langle x \rangle$  is a maximal ideal of  $Q[x]$ .

- ① Show that  $f(x) = 8x^3 + 6x^2 - 9x + 24$  is irreducible over  $\mathbb{Q}$ .
- $$f(x) = 24 - 9x + 6x^2 + 8x^3$$
- here  $a_0 = 24$ ,  $a_1 = -9$ ,  $a_2 = 6$ ,  $a_3 = 8$   
 let us take the prime number 3 so that  
 $3|24$ ,  $3|24$ ,  $3|(-9)$ ,  $3|6$  but  $3 \nmid 8$

hence by Eisenstein criterion of irreducibility,  $f(x)$  is irreducible over  $\mathbb{Q}$ .

- ② Find the principal ideal generated by 5 in the ring of integers  $(\mathbb{Z}, +, \cdot)$ .

here  $\mathbb{Z}$  is a commutative ring with unity so that  $(5)$  contains all the element of the form  $5r$ , where  $r \in \mathbb{Z}$

Therefore, the principal ideal generated by 5 is given by

$$5 = \{ \dots, -15, -10, -5, 0, 5, 10, 15, \dots \}$$

It is evident that  $(-5) = 5$

- ③ Determine whether the polynomial  $x^3 + 5x^2 + 3x - 9$  has zeroes of multiplicity greater than 1. They exist, find all zeroes.

Let  $f(x) = x^3 + 5x^2 + 3x - 9$ .

Evidently,  $f(1) = 0$ . Therefore,  $x = 1$  is a zero of  $f(x)$ . Dividing  $f(x)$  by  $x - 1$ , we get  $x^2 - 6x + 9$ , i.e.  $(x - 3)^2$ .

Thus  $x = 1, -3, -3$  are the zeros of  $f(x)$ . Clearly,  $x = -3$  is a zero of multiplicity greater than 1.

(4) In the integral domain  $(\mathbb{Z}[i] : a, b \in \mathbb{Z}, +, -)$  of Gaussian integers.

$(2+3i) \mid (-6+11i)$ ;

Let  $-6+11i = (2+3i)(x+iy)$   $\Rightarrow$   
 $= (2x-3y) + i(3x+2y) \dots (1)$

Equating real & imaginary parts on both sides, we get

Solving these eq<sup>n</sup>, we get  $x = 7$   
 $2x - 3y = -6$   
 $\& \quad 3x + 2y = 11$

$$\begin{array}{r} 4x - 6y = -12 \\ 9x + 6y = 51 \\ \hline 13x = 39 \\ \boxed{x = 3} \end{array}$$

from eq<sup>n</sup>  
 $2 \times 3 - 3y = -6$   
 $6 + 6 = 3y$   
 $\boxed{y = 4}$

using these values in (1), we get find that  
 $(-6+17i) = (2+3i)(3+4i)$   
 and so,  $(2+3i) \mid (-6+17i)$

(5) Find all the units of integral domain  $(\{a+ib: a, b \in \mathbb{Z}\}, +, \cdot)$  of Gaussian integers.

Let  $u+iv$  be a unit and  $u'+iv'$  its inverse. Then

$$(u+iv)(u'+iv') = 1$$

$$\text{i.e., } (uu' - vv') + i(uv' + vu') = 1$$

Equating real & imaginary parts, this gives

$$uu' - vv' = 1$$

$$\& \quad uv' + vu' = 0$$

Squaring (1) & (2) and then adding the resulting eq<sup>n</sup>, we get

$$u^2v'^2 + v^2v'^2 + u^2v'^2 + v^2v'^2 = 1$$

$$\text{i.e., } (u^2+v^2)(u'^2+v'^2) = 1$$

Now the product of two positive integers can be 1 only when each of them

is 1. Hence

$$u^2 + v^2 = 1$$

This gives  $u^2 = 1$   $v^2 = 0$   
or  $u^2 = 0$ ,  $v^2 = 1$

i.e.,  $u^2 = \pm 1$   $v = 0$

$$\text{or } u = 0 \quad v = \pm 1$$

Thus the required units are  $\pm 1, \pm i, \pm j$

(6) Let  $S$  be the set of all ordered pairs of real numbers. Define sums & scalar multipliers pairs as follows

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\& c(x, y) = (cx, y)$$

Show that  $S$  is not a vector space.

Pf  $(x, y) \in S$  &  $c, d \in \mathbb{R}$ , then

$$(c+d)(x, y) = ((c+d)x, y) = (cx + dx, y)$$

$$c(x, y) + d(x, y) = (cx, y) + (dx, y) = (cx + dx, y)$$

Thus  $(c+d)(x, y) \neq c(x, y) + d(x, y)$   
unless  $y = 0$

Hence the set  $S$  is not a vector space

(7) If  $V(F)$  is a vector space, then prove that  
 $y + (x - y) = x, \forall x, y \in V$

we have

$$y + (x - y) = y + (x + (-y))$$

$$= y + ((-y) + x), \quad \text{by commutativity of } + \text{ in } V$$

$$= (y + (-y)) + x, \quad \text{by associativity of } + \text{ in } V$$

$$0 + x, \quad \text{since } y + (-y) = 0$$

$x$ , since  $0$  is the additive identity in  $V$ .

(8) Prove that vectors  $(1, 1, 0)$ ,  $(3, 1, 3)$  &  $(5, 3, 3)$  are linearly dependent.

Let  $\alpha_1, \alpha_2, \alpha_3$  be any three scalars in  $R$ . Then

$$\alpha_1(1, 1, 0) + \alpha_2(3, 1, 3) + \alpha_3(5, 3, 3) = 0, 0, 0$$

$\therefore$  the zero vector in  $R$

$$\Rightarrow (\alpha_1, \alpha_1, 0) + (3\alpha_2, \alpha_2, 3\alpha_2) + (5\alpha_3, 3\alpha_3, 3\alpha_3) = 0, 0, 0$$

$$(\alpha_1 + 3\alpha_2 + 5\alpha_3, \alpha_1 + \alpha_2 + 3\alpha_3, 3\alpha_2 + 3\alpha_3) = (0, 0, 0)$$

$$\Rightarrow \alpha_1 + 3\alpha_2 + 5\alpha_3 = 0$$

$$\alpha_1 + \alpha_2 + 3\alpha_3 = 0$$

$$+ 3\alpha_2 + 3\alpha_3 = 0$$

One of the non-trivial solutions of these  
 $\alpha_1 = 6, \alpha_2 = 3, \alpha_3 = -3$

Thus we have

$$6(1, 1, 0) + 3(3, 1, 3) + (-3)(5, 3, 3) = (0, 0, 0)$$

Hence the vectors  $(1, 1, 0), (3, 1, 3)$  &  $(5, 3, 3)$   
 are linearly dependent.

(9) Show that the set  $(1+i, 1-i)$  is a basis  
 of the vector space  $C(R)$

Let  $S = \{1+i, 1-i\}$ . The set  $S$  is L.P.

Now let  $x+iy$  be any element in  $V$ . Then

$$x+iy = \alpha_1(1+i) + \alpha_2(1-i)$$

$$= (\alpha_1 + \alpha_2) + i(\alpha_1 - \alpha_2)$$

$$\Rightarrow \alpha_1 + \alpha_2 = x \quad \& \quad \alpha_1 - \alpha_2 = y$$

$$\alpha_1 = \frac{1}{2}(x+y) \in R \quad \& \quad \alpha_2 = \frac{1}{2}(x-y) \in R$$

Thus every vector in  $C$  can be expressed  
 as a linear combination of vectors  
 in  $S$   
 hence  $S$  is a basis for  $C(R)$

(10) Show that the vectors  $1+i$ ,  $1-i$  are linearly independent in  $\mathbb{C}(\mathbb{R})$  but linearly dependent in  $\mathbb{C}(\mathbb{C})$

Let  $\alpha_1, \alpha_2$  be any two real numbers. Then

$$\alpha_1(1+i) + \alpha_2(1-i) = 0$$

$$\Rightarrow (\alpha_1 + \alpha_2) + i(\alpha_1 - \alpha_2) = 0$$

$$\Rightarrow \alpha_1 + \alpha_2 = 0, \quad \alpha_1 - \alpha_2 = 0$$

$$\alpha_1 = \alpha_2 = 0$$

hence, the given vectors are linearly independent in  $\mathbb{C}(\mathbb{R})$ .

Further, let  $\alpha_1, \alpha_2$  be any two complex numbers such that

$$\alpha_1 = x_1 + iy_1 \quad \& \quad \alpha_2 = x_2 + iy_2$$

where  $x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$\text{Then } \alpha_1(1+i) + \alpha_2(1-i) = 0$$

$$(x_1 + iy_1)(1+i) + (x_2 + iy_2)(1-i) = 0$$

$$\Rightarrow (x_1 - y_1 + x_2 + y_2) + i(x_1 + y_1 - x_2 + y_2) = 0$$

Now,  $x_1 = 1, y_1 = 1$  &  $x_2 = 1, y_2 = -1$  this eq<sup>n</sup> is satisfied. But in this case, we have

$$\alpha_1 = 1+i \neq 0 \quad \& \quad \alpha_2 = 1-i \neq 0$$

which shows that the given vectors are independent in  $\mathbb{C}(\mathbb{C})$