

Tangent And Normals

1. Eq. of tangent and normal at any Pt. (x_1, y_1) .

$$\Rightarrow y - y_1 = \frac{dy_1}{dx_1} (x - x_1) \rightarrow \text{Eq. of tangent}$$

$$y - y_1 = -\frac{dx_1}{dy_1} (x - x_1) \rightarrow \text{Eq. of Normal}$$

2. Find the Eq. of tangent and normal to the curve $y^2 = 3x^2 + 1$ at the Pt. $(1, 2)$.

Sol.

$$y^2 = 3x^2 + 1$$

$$2y \frac{dy}{dx} = 6x \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$$

at Pt. $(1, 2)$ $\frac{dy}{dx} = \frac{3 \times 1}{2} = \frac{3}{2} \Rightarrow$ slope of the tangent is $\frac{3}{2}$.

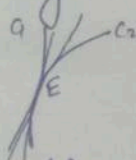
$$\Rightarrow \text{Eq. of tangent is } y - 2 = \frac{3}{2} (x - 1).$$

Now slope of Normal is $-\frac{2}{3}$.

$$\Rightarrow \text{Eq. of Normal is } y - 2 = -\frac{2}{3} (x - 1)$$

$$\text{i.e. } 2x + 3y - 8 = 0.$$

3. Angle of Intersection of two curves.



$$\left. \begin{array}{l} y = m_1x + c_1 \\ y = m_2x + c_2 \end{array} \right\} \rightarrow \text{eq. of tangents}$$

i.e. the angle, b/w the tangents at their common point of intersection E.

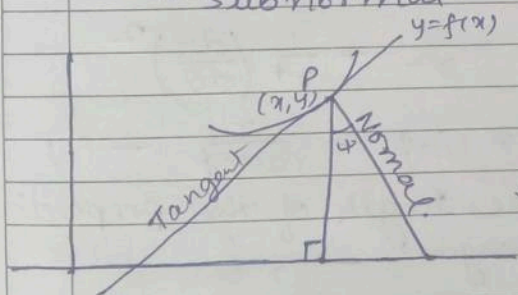
angle b/w these lines $\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

Here m_1 & m_2 are the values of dy/dx for two curves at E.

Note - * If two curve touch each other then $m_1 = m_2$

* If two curve cut orthogonally then $m_1 m_2 = -1$.

4. Length of the tangent, subtangent, Normal & Subnormal.



$$\text{Tangent} \rightarrow y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\text{sub tangent} \rightarrow y \frac{dx}{dy}$$

$$\text{Normal} \rightarrow y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\text{Sub Normal} \rightarrow y \cdot \frac{dy}{dx}$$

5. In the catenary $y = a \cosh(x/a)$ find out length of the portion of the normal intercepted b/w the curve & the axis of x.

$$y = a \cosh x/a \Rightarrow \frac{dy}{dx} = \sinh x/a$$

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$$\text{length of Normal } y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{--- (1)}$$

$$= y \sqrt{1 + \sinh^2 x/a} = y \cosh x/a$$

$$= y(y/a) = y^2/a \quad \text{Ans}$$

6. Length of tangent, Normal, subtangent & subNormal in polar co-ordinates.

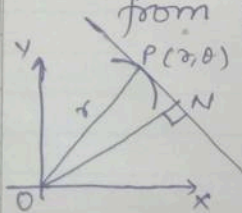
Tangent $r \sqrt{1 + \left(r \frac{d\theta}{dr}\right)^2}$

Normal $\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

subtangent $r^2 \frac{d\theta}{dr}$, subnormal $\frac{dr}{d\theta}$

Ques 7. Prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2$

sol we know that $p = r \sin \phi = \frac{r}{\operatorname{cosec} \phi}$ — (1)
where p be the length of the perpendicular from pole O to the tangent.



eq. (1) $p = \frac{r}{\operatorname{cosec} \phi} = \frac{r}{\sqrt{1 + \cot^2 \phi}} = \frac{r}{\sqrt{1 + \left(\frac{1}{r} \frac{dr}{d\theta}\right)^2}}$

$$\Rightarrow \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} \Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2$$

Asymptotes (अनंत स्पर्शी)

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Q. An infinite branch of a curve is said to be Asymptote.

$\Rightarrow$ Asymptote $\parallel$ to x -axis.

Ques - Curve $x^2y - 3x^2 - 5xy + 6y + 2 = 0$. find asymptotes $\parallel$ to x axis.

Sol. Arrange the given eq.

$$(y-3)x^2 - 5xy + 6y + 2 = 0$$

Equating to zero the coefficient of highest degree term i.e. x^2

$$\Rightarrow y-3=0 \Rightarrow y=3 \text{ required asymptote } \parallel \text{ to } x \text{ axis.}$$

Note * Similarly we can do for $\parallel$ to y axis.

i.e. equating to zero the coefficient of highest degree term of y .

* n^{th} degree Eq. have n max. no. of asymptotes.

Q. Oblique Asymptotes -

Put $y = mx + c$. & $m = \lim_{x \rightarrow \infty} y/x$.
 $\phi_m(m)$ by putting $x=1$ & $y=m$.

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Note - $\phi_n'(m)=0$ but $\phi_{n-1}(m) \neq 0$ then there is no asymptotes.

10. Find all asymptotes of the curve

$$2x^3 + 3x^2y - 2xy^2 - 3y^3 + 6x + y + 3 = 0.$$

Sol $\rightarrow$ Given curve is of third degree.

Putting $x=1$ & $y=m$ we get

third degree term. $\phi_3(m) = 2 + 3m - 2m^2 - 3m^3$

$$\phi_3'(m) = 3 - 4m - 9m^2$$

Second degree term.

$$\phi_2(m) = 0$$

first degree term. $\phi_1(m) = 6 + m$.

Slope of asymptotes are the roots of $\phi_3(m) = 0$.

$$2 + 3m - 2m^2 - 3m^3 = 0. \Rightarrow m = -2/3, -1, 1$$

$$C = -\frac{\phi_2(m)}{\phi_3'(m)} = 0 \quad (\text{since } \phi_2(m) = 0).$$

$\Rightarrow$ asymptotes $y = -2/3x$, $y = -x$, $y = x$.

Curvature - Def. curvature of straight line is zero. because

11 there is no bending. Radius of curvature $\rho = \frac{ds}{d\psi}$.

Ques Find the radius of curvature of the curve

$$s = 4a \sin \psi$$

Sol:- diff. the given cycloid w.r.t ψ we get

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$$\frac{ds}{d\psi} = 4a \cos \psi$$

$$\Rightarrow \rho = 4a \cos \psi \quad \text{required radius of curvature.}$$

For Cartesian curves

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

12 Envelope and Evolutes.

1. Def - The locus of the pt. of intersection of two neighbouring curves of a family is called the envelope of the family of curves.

2. Def - The locus of centres of curvature of a curve is called its evolute and that curve is said to be an involute of evolute.

13 Find the envelope of the curve

$$y = \alpha x + \frac{3}{2\alpha} \quad \alpha \text{ being a parameter}$$

Sol - $y = \alpha x + \frac{3}{2\alpha} \Rightarrow 2\alpha y = 2\alpha^2 x + 3 \quad \text{--- (1)}$

diff w.r.t α . we get $2y = 4\alpha x \Rightarrow \alpha = \frac{y}{2x}$

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Put the value of α in eq. (1)

$$2\alpha y = 2\alpha^2 x + 3$$

$$\frac{y^2}{x} = \frac{y^2}{2x} + 3 \Rightarrow \frac{y^2}{2x} = 3$$

$$\Rightarrow y^2 = 6x$$

This is the required eq. of the envelope.

14. Concavity, convexity and Pt. of inflexion
* Test for convexity with respect to x-axis.

If $y \frac{d^2y}{dx^2} > 0 \Rightarrow$ curve is convex at P. w.r.t x-axis.

* If $y \frac{d^2y}{dx^2} < 0 \Rightarrow$ curve is said to be concave at P. w.r.t x-axis.

similarly.

$x \frac{d^2x}{dy^2} > 0 \Rightarrow$ convex ^{at P} w.r.t y-axis

$x \frac{d^2x}{dy^2} < 0 \Rightarrow$ concave at P w.r.t y-axis

15 The curve $y = e^x$ convex at every Pt. w.r.t the foot of corresponding ordinate

sol -

$$y = e^x \Rightarrow \frac{dy}{dx} = e^x \Rightarrow \frac{d^2y}{dx^2} = e^x$$

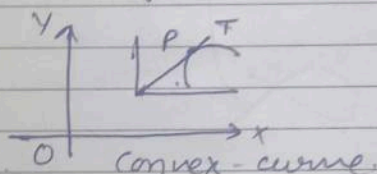
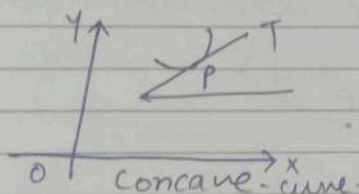
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$$y \frac{d^2y}{dx^2} = e^x = e^x \cdot e^x$$

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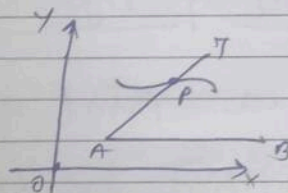
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$\Rightarrow y \frac{d^2y}{dx^2} > 0 \Rightarrow$ curve is convex at every pt. w.r.t the foot of corresponding ordinate.



16. Point of inflexion

$$\frac{d^2y}{dx^2} = 0 \quad \& \quad \frac{d^3y}{dx^3} \neq 0$$



17. Def Singular points - This is an unusual pt. on a curve.

Multiple point - A point through which a curve passes more than one time.

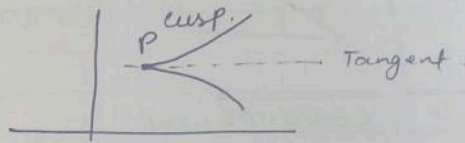
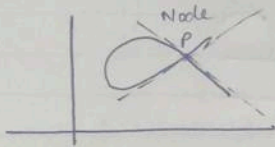
Double point - If a curve passes two times through a point then that pt. is known as double pt.

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Double Point —
 $\begin{array}{l} \rightarrow \text{Node} \\ \rightarrow \text{cusp.} \end{array}$

Note — A double pt. at which two real tangents can be drawn.

cusp — A double pt. is called cusp if the two tangents at it are coincident.



Point of inflexion — A point where the curve crosses the tangent is called a point of inflexion.

Condition for symmetry —

- * A curve is symmetric about x -axis if eq. remains same by replacing y by $-y$.

for ex — $y^2 = 4ax$.

Similarly $x^2 = 4ay$ symmetric about y -axis.

* Symmetric about the line $y = x$ on interchanging x & y . for example $x^3 + y^3 = 3axy$.

* Symmetric in the opposite quadrants if its equation remains the same when x & y replaced by $-x$ and $-y$ respectively.

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*. curve through origin - If the eq. of curve doesn't contain constant term.

*. eq. of tangent at the origin can be obtained by equating to zero the lowest degree term in the eq. of the curve.

*. By putting $y=0$ in eq. of a curve. we get point of intersection with x-axis. Similarly putting $x=0$ " " " " with y-axis.

*. Regions in which the curve does not lie -

If the value of y is imaginary for certain value of x then the curve does not exist for such values.

Ex $y^2 = 4x$

for -ve value of x , y is imaginary

$\Rightarrow$ no curve in second & third quadrants

Remark - POSTAR

P $\rightarrow$ Point of intersection, O $\rightarrow$ origin

S $\rightarrow$ symmetry, T $\rightarrow$ Tangent, A $\rightarrow$ Asymptotes

R - Region

same as for polar co-ordinates.

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Def. of seq.

Types of seq.

Def. of convergent seq.

(ϵ, δ) definition.

with examples

19

Necessary condition for cgt series is

$$\lim_{n \rightarrow \infty} u_n = 0.$$

Cauchy's Fundamental test for Divergence.

if $\lim_{n \rightarrow \infty} u_n \neq 0$ the series is divergent.

* P-series test. $\sum \frac{1}{n^p}$ cgt for $p > 1$.

* Leibnitz test for alternating series.

(a) monotone decreasing.

(b) alternate +ve & -ve.

(c) $\lim_{n \rightarrow \infty} u_n = 0 \Rightarrow \sum u_n$ cgt.

20

(ϵ, δ) definition for limit & continuity.

for a given $\epsilon > 0 \exists \delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - l| < \epsilon.$$

Cauchy
Def of
continuity

for a given $\epsilon > 0 \exists \delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

21. Boundedness theorem -

If the fun. is continuous in the closed interval $[a, b]$ then f is bdd in this interval.

Borel's Theo -

If the fun. f is continuous on the closed interval $[a, b]$ then the interval can always be divided up into a finite no. of subintervals : for a given $\epsilon > 0$

$|f(x_1) - f(x_2)| < \epsilon$ where $x_1 \neq x_2$ are any two points in the same subinterval.

Heine's definition for continuity -

A fun $f: D \rightarrow R$ is said to be continuous at $c \in D$ if for every seq. $\langle x_n \rangle$ in D that converges to c , the seq. $\langle f(x_n) \rangle \rightarrow f(c)$.

22

Evaluate $\lim_{x \rightarrow 0} \frac{\log \sin 2x}{\log \sin x}$.

$$\begin{aligned} \text{Sol} \rightarrow \lim_{x \rightarrow 0} \frac{\log \sin 2x}{\log \sin x} &= \lim_{x \rightarrow 0} \frac{\log \frac{2 \cos 2x}{\sin 2x}}{\frac{\cos x}{\sin x}} \\ &= \lim_{x \rightarrow 0} \frac{2 \cot 2x}{\cot x} = \lim_{x \rightarrow 0} \frac{2 \tan x}{\tan 2x} = \lim_{x \rightarrow 0} \frac{\tan x}{2 \tan x} \\ &= 1. \end{aligned}$$

23Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$

Sol $\rightarrow$ $\lim_{x \rightarrow 0} \frac{\left(1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right) - 1 - x}{x^2} \quad \left[\frac{0}{0}\right]$

$$= \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + \frac{x^3}{6} + \dots}{x^2} = \lim_{x \rightarrow 0} \left(\frac{1}{2} + \frac{x}{6} + \dots\right)$$

$$= \frac{1}{2}$$

24Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Sol $\rightarrow$ $\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x}$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{1+1+0} = 0$$

25 $\lim_{x \rightarrow \pi/2} (\cos x)^{\cos x}$ Sol Taking log both side.

$$y = (\cos x)^{\cos x} \Rightarrow \log y = \cos x \log(\cos x)$$

$$\log y = \lim_{x \rightarrow \pi/2} (\cos x \log \cos x) = \lim_{x \rightarrow \pi/2} \frac{\log \cos x}{\sec x}$$

$$\log y = \lim_{x \rightarrow \pi/2} \frac{-\sin x}{\sec x \tan x} = -\lim_{x \rightarrow \pi/2} \cos x = 0$$

$$\Rightarrow y = e^0 = 1$$

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26

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(ii)

(iii)

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27

(i)

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26 Rolle's Theorem —

if a fun. $f(x)$ is

(i) continuous in the closed interval $[a, b]$

(ii) Differentiable in the open interval (a, b)

(iii) $f(a) = f(b)$.

then there exist a point $c \in (a, b)$ such that

$$f'(c) = 0$$

Ex

Rolle's theo. applicable for $f(x) = 3x^4 - 4x^2 + 5$
in $[-1, 1]$

Rolle's theo. not applicable for $f(x) = |x|$
in the interval $[-1, 1]$.

because $|x|$ is not differentiable at $x = 0$.

27 Lagrange's Mean-Value Theorem

(i) conti in closed interval $[a, b]$

(ii) diff in open interval (a, b) .

then there exist at least one pt. $c \in (a, b)$

$$: \frac{f(b) - f(a)}{b - a} = f'(c)$$

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Do some examples.

28 If $f(a+h)$ can be expanded in a series of ascending powers of h then the expansion is

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \dots$$

Put $h = x - a$.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \dots$$

* Do some questions for practice. (two or three)

29 Maclaurin's Series -

Let $f(x)$ be a continuous fun. and its derivative upto n^{th} and higher order exist then putting $a = 0$ in Taylor's Series, we get.

$$f(x) = f(0) + \frac{x}{1} f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3} f'''(0) + \dots$$

* Do one or two relatable questions from your book.

Ex (1) $y = e^x$.

(2) $y = \sin^{-1} x$.

30 Euler's Theorem -

If z is a homogeneous fun. of x, y of degree n and $z = f(u)$ then.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

* Homo. Func $\rightarrow a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$ is a homo.

Expt. No. fun. of order n.

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3) Euler's Theo. on Homogeneous Func.

if z is a homogeneous fun. of x, y of order n then

$$x \cdot \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n z.$$

Integral Calculus

1. Riemann Integral - The area A bounded by the curve $y=f(x)$, x -axis and the ordinates $x=a$ and $x=b$ is more than lower sum and less than upper sum.

Lower sum $\leq$ Area of the curve $\leq$ upper sum.

$$L(f, P) \leq A \leq U(f, P).$$

2. Fundamental Theorem of Integral Calculus.

First - Let f be a continuous fun. on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$ for all $x \in [a, b]$.

Second - Let f be continuous fun. defined on the closed interval $[a, b]$ and ϕ be an anti-derivative of f . Then $\int_a^b f(x) dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$.

Ex ① $y = e^x$. where $A(x) = \int_a^x f(x) dx$.

30 Euler's Theorem -

if z is a homogeneous fun. of x, y of degree n and $z = f(u)$ then.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}.$$

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3. Evaluate $\int_1^2 \frac{1}{x(1+\log x)^2} dx$.

Sol. Let $I = \int_1^2 \frac{1}{x(1+\log x)^2} dx$.

Putting $1+\log x = t \Rightarrow \frac{1}{x} dx = dt$.

For lower limit $x=1 \Rightarrow t = 1+\log x = 1+\log 1 = 1+0 = 1$

For upper limit $x=2 \Rightarrow t = 1+\log x = 1+\log 2$

$$I = \int_1^{1+\log 2} \frac{dt}{t^2} = -\left[\frac{1}{t}\right]_1^{1+\log 2}$$

$$= -\left[\frac{1}{1+\log 2} - 1\right] = \frac{\log 2}{1+\log 2} \quad \underline{\text{Ans}}$$

4. Evaluate $\int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x} dx$.

Sol. Let $I = \int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x} dx = \int_{\pi/6}^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$.

Putting $\sin x = t \Rightarrow \cos x dx = dt$.

For lower limit $x = \pi/6 \Rightarrow t = \sin x = \sin \pi/6 = 1/2$

For upper limit $x = \pi/2 \Rightarrow t = \sin x = \sin \pi/2 = 1$

$$I = \int_{1/2}^1 \frac{dt}{1+t^2} = \left[\tan^{-1} t \right]_{1/2}^1$$

$$= \tan^{-1} 1 - \tan^{-1} 1/2 = \tan^{-1} \left[\frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} \right] = \tan^{-1} 1/3 \quad \underline{\underline{\text{Ans}}}$$

5. Evaluate $\int_1^2 |x^3 - x| dx$.

Sol. we note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and $x^3 - x \geq 0$ on $[1, 2]$

So we can write.

$$\int_1^2 |x^3 - x| dx = \int_{-1}^0 (x^3 - x) dx + \int_0^1 -(x^3 - x) dx + \int_1^2 (x^3 - x) dx$$

$$= \int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx + \int_1^2 (x^3 - x) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_1^2$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = 11/4 \quad \underline{\underline{\text{Ans}}}$$

6. Evaluate $\int_{0.2}^{3.5} [x] dx$.

Sol. $[x]$ is the greatest integer fun.

$$f(x) = [x] = \begin{cases} 0 & 0.2 \leq x \leq 1 \\ 1 & 1 \leq x \leq 2 \\ 2 & 2 \leq x \leq 3 \\ 3 & 3 \leq x \leq 3.5 \end{cases}$$

$$\int_{0.2}^{3.5} [x] dx = \int_{0.2}^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx + \int_3^{3.5} 3 dx = 0 + 1 + 2 + 1.5 = 4.5$$

Ans7.

$$\int_{-\pi/2}^{\pi/2} f(x) dx \quad \text{where } f(x) = \sin|x| + \cos|x|.$$

Sol.

$$f(x) = \sin|x| + \cos|x|.$$

$$f(-x) = \sin|-x| + \cos|-x| = \sin|x| + \cos|x| = f(x).$$

$$\int_{-\pi/2}^{\pi/2} f(x) dx = \int_{-\pi/2}^0 \{\sin|x| + \cos|x|\} dx + \int_0^{\pi/2} \{\sin|x| + \cos|x|\} dx$$

$$I = 2 \int_0^{\pi/2} f(x) dx = 2 \int_0^{\pi/2} [\sin x + \cos x] dx$$

$$= 2 [-\cos x + \sin x]_0^{\pi/2}$$

$$= 2 [-(\cos \pi/2 - \cos 0) + (\sin \pi/2 - \sin 0)]$$

$$= 2 [-(0-1) + (1-0)] = 4$$

Ans8.

$$\text{Prove that } \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx = 0.$$

Sol

$$\text{using theorem } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2}-x) - \cos(\frac{\pi}{2}-x)}{1 + \sin(\frac{\pi}{2}-x) \cos(\frac{\pi}{2}-x)} dx$$

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$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \Rightarrow 2I = 0 \Rightarrow I = 0$$

9 Evaluate $\int_0^{\pi} \sin^6 x \, dx$.

Sol we know that $\int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx$ if $f(2a-x) = f(x)$.

$$f(\pi-x) = \sin^6(\pi-x) = \sin^6 x = f(x).$$

$$I = 2 \int_0^{\pi/2} \sin^6 x \, dx.$$

$$= 2 \cdot \frac{5 \times 3 \times 1}{6 \times 4 \times 2} \cdot \frac{\pi}{2}$$

$$\text{Since } \int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\dots}{n(n-2)(n-4)} \times \frac{\pi}{2} \Big]$$

$$= \frac{5\pi}{16} \quad \underline{\underline{\text{Ans}}}$$

10 Evaluate $\int_{-\pi}^{\pi} (1-x^2) \sin x \cos^2 x \, dx$.

Sol - $I = \int_{-\pi}^{\pi} (1-x^2) \sin x \cos^2 x \, dx.$

$$\text{Here } f(-x) = -f(x)$$

$$\Rightarrow I = 0.$$

Mean Value Theorem of Integral Calculus

11. Mean Value of $y = \frac{1}{b-a} \int_a^b y \, dx$.

Ques - Find the mean value of the ordinates of $y^2 = 4x$ from $(0,0)$ to $(4,4)$ uniformly along the x -axis.

Sol - Mean Value $\frac{\int_0^4 y \, dx}{4-0}$

$$= \frac{\int_0^4 \sqrt{4x} \, dx}{4} = \frac{2}{4} \int_0^4 x^{1/2} \, dx = \frac{1}{2} \cdot \frac{2}{3} (x^{3/2})_0^4$$

$$= \frac{1}{3} (8) = 8/3 \quad \underline{\text{Ans}}$$

12. Find the mean value of $x(a-x)$ over the range $(0, a)$.

Sol. Mean Value $\frac{\int_0^a x(a-x) \, dx}{a-0}$

$$= \frac{1}{a} \left[\frac{ax^2}{2} - \frac{x^3}{3} \right]_0^a = \frac{1}{a} \left[\frac{a^3}{2} - \frac{a^3}{3} \right]$$

$$= \frac{1}{a} \cdot \frac{a^3}{6} = a^2/6$$

Differentiation under Integrals -

13.

if $f(x, \alpha)$ and $\frac{\partial f(x, \alpha)}{\partial \alpha}$ be continuous fun. of x and α , then.

$$\frac{d}{d\alpha} \left[\int_a^b f(x, \alpha) dx \right] = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx.$$

Que - $\int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)} dx.$

Sol $\rightarrow \frac{dI}{da} = \int_0^{\infty} \frac{\partial}{\partial a} \frac{\tan^{-1}(ax)}{x(1+x^2)} dx$

$$= \int_0^{\infty} \frac{1}{x(1+x^2)} \cdot \frac{x}{1+a^2x^2} dx = \int_0^{\infty} \frac{1}{(1+x^2)(1+a^2x^2)} dx$$

do partial fractions,

$$= \int_0^{\infty} \frac{1}{1-a^2} \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2} \right] dx$$

$$= \frac{1}{1-a^2} \left[\tan^{-1}x - a \tan^{-1}ax \right]_0^{\infty}.$$

$$= \frac{1}{1-a^2} \left[\frac{\pi}{2} - a \frac{\pi}{2} \right] = \frac{\pi}{2} \frac{(1-a)}{1-a^2} = \frac{\pi}{2} \cdot \frac{1}{1+a}$$

$$I = \frac{\pi}{2} \log(1+a) + c \quad \text{when } a=0 \Rightarrow I=0 \Rightarrow c=0$$

Hence $I = \frac{\pi}{2} \log(1+a)$

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Que 14

$$\frac{d}{da}$$

Que

Sol.

$$\frac{dI}{da} =$$

$$=$$

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Rule of Differentiation under the Integral Sign

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Ques 14.

$$\frac{d}{d\alpha} \left\{ \int_{\phi(\alpha)}^{\psi(\alpha)} f(x, \alpha) dx \right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x, \alpha)}{\partial \alpha} dx - \frac{d\phi}{d\alpha} f[\phi(\alpha), \alpha] + \frac{d\psi}{d\alpha} f[\psi(\alpha), \alpha] \quad \text{--- (*)}$$

Ques Evaluate $\int_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} \frac{\sin ax}{x} dx$.

Sol. using (*) eq. we can write.

$$\begin{aligned} \frac{dI}{da} &= \int_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} \frac{\partial}{\partial a} \left(\frac{\sin ax}{x} \right) dx - \frac{d}{da} \left(\frac{\pi}{6a} \right) \cdot \frac{\sin a \cdot \frac{\pi}{6a}}{\frac{\pi}{6a}} + \frac{d}{da} \left(\frac{\pi}{2a} \right) \cdot \frac{\sin a \cdot \frac{\pi}{2a}}{\frac{\pi}{2a}} \\ &= \int_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} \cos ax dx + \frac{1}{2a} - \frac{1}{a} = \left[\frac{\sin ax}{a} \right]_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} - \frac{1}{2a} \\ &= \frac{1}{a} \left[\sin \frac{\pi}{2} - \sin \frac{\pi}{6} \right] - \frac{1}{2a} = \frac{1}{a} \left[1 - \frac{1}{2} \right] - \frac{1}{2a} = \frac{1}{2a} - \frac{1}{2a} = 0 \end{aligned}$$

Ans

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Convergence of Improper Integrals

Ques 15.

Test the convergence of $\int_0^{\infty} \frac{dx}{1+x^2}$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b \rightarrow \infty} [\tan^{-1} x]_0^b = (\tan^{-1} \infty - \tan^{-1} 0) = \pi/2$$

which is finite $\Rightarrow$ Integral is cgt.
Ans

Ques 16.

Test the convergence of $\int_0^{\infty} \frac{dx}{1+x}$

Sol.

$$\int_0^{\infty} \frac{dx}{1+x} = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x}$$

$$= \lim_{b \rightarrow \infty} [\log(1+x)]_0^b$$

$$= \lim_{b \rightarrow \infty} [\log(1+b) - \log 1] = [\log \infty - 0] = \infty$$

$\Rightarrow$ Integral is divergent.

Ques 17.

Test the convergence of the $\int_0^{\infty} \cos x dx$

$$\int_0^{\infty} \cos x dx = \lim_{b \rightarrow \infty} \int_0^b \cos x dx = \lim_{b \rightarrow \infty} [\sin x]_0^b = \lim_{b \rightarrow \infty} [\sin b - \sin 0]$$

$$= (\sin \infty - 0) = \sin \infty \text{ limit doesn't exist}$$

$\Rightarrow$ Integral is oscillatory.

Ques 18. Learn def. of Absolute, Conditional convergence.

Learn Comparison test for convergence

Ex Check the convergence of the integral

Ques $\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$

Sol $f(x) = \frac{\sin^2 x}{x^2} dx \Rightarrow f(x) = \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} g(x)$

$\int_1^{\infty} g(x) dx = \int_1^{\infty} \frac{1}{x^2} dx$ is cgt since $p=2 > 1$

by using comparison test $\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$ is cgt

Ques 19. $\int_0^{\infty} \frac{dx}{a+x^2}$ and $\int_0^{\infty} \frac{x^2 dx}{a^2+x^2}$

if $p > 1$ then integral is cgt.
 $p \leq 1$ " " " " divergent.

Ques 20 check the convergence of $\int_4^{\infty} \frac{3x+8}{x^4+7} dx$

Sol - $f(x) = \frac{3x+8}{x^4+7} < \frac{3x}{x^4} \sim \frac{3}{x^3} = 3g(x)$

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$\int_0^{\infty} g(x) dx = \int_0^{\infty} \frac{1}{x^3} dx$ is convergent by
p-integral ($p=3$)

$\Rightarrow \int_4^{\infty} \frac{3x+8}{x^4+7}$ is convergent.

Ques 21. (i) If $\frac{d\vec{a}}{dt} = \vec{c} \times \vec{a}$, $\frac{d\vec{b}}{dt} = \vec{c} \times \vec{b}$ Show that

$$\frac{d}{dt} (\vec{a} \times \vec{b}) = \vec{c} \times (\vec{a} \times \vec{b}).$$

Sol - $\frac{d}{dt} (\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$
 $= -\{(\vec{b} \cdot \vec{a}) \vec{c} - (\vec{b} \cdot \vec{c}) \vec{a}\} + \{(\vec{a} \cdot \vec{b}) \vec{c} - (\vec{a} \cdot \vec{c}) \vec{b}\}$
 $= (\vec{b} \cdot \vec{c}) \vec{a} - (\vec{a} \cdot \vec{c}) \vec{b} = (\vec{c} \cdot \vec{b}) \vec{a} - (\vec{c} \cdot \vec{a}) \vec{b} = \vec{c} \times (\vec{a} \times \vec{b})$

⁽²⁾
Ques (ii) If $\vec{r} \times d\vec{r} = \vec{0}$ show that $\hat{r} = \text{constant}$

Sol - $|\vec{r}| = r$ so that $\vec{r} = r \hat{r}$

therefore $d\vec{r} = d(r \hat{r}) = dr \hat{r} + r d\hat{r}$

$$\vec{r} \times d\vec{r} = (r \hat{r}) \times (dr \hat{r} + r d\hat{r}) = r^2 \hat{r} \times d\hat{r}$$

Since $\hat{r} \times \hat{r} = 0$ whence $\vec{r} \times d\vec{r} = 0$

$$\Rightarrow r^2 \hat{r} \times d\hat{r} = \vec{0} \Rightarrow \hat{r} \times d\hat{r} = 0 \quad \text{--- (i)}$$

Since $\hat{r}$ is of constant modulus

we have $\hat{r} \cdot d\hat{r} = 0$ --- (ii)

using (i) and (ii) $d\hat{r} = 0 \Rightarrow \hat{r}$ is a constant vector.

Vector Calculus

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Ques 22 If a, b are constant vectors, ω is a constant and $\vec{r}$ is a vector fun. of scalar variable t given by $\vec{r} = \cos \omega t \vec{a} + \sin \omega t \vec{b}$ show that.

(i) $\frac{d^2 \vec{r}}{dt^2} + \omega^2 \vec{r} = 0$

(ii) $\vec{r} \times \frac{d\vec{r}}{dt} = \omega \vec{a} \times \vec{b}$

Sol - (i) $\frac{d\vec{r}}{dt} = -\omega \sin \omega t \vec{a} + \omega \cos \omega t \vec{b}$

$$\frac{d^2 \vec{r}}{dt^2} = \frac{d}{dt} \left(\frac{d\vec{r}}{dt} \right) = \frac{d}{dt} (-\omega \sin \omega t \vec{a} + \omega \cos \omega t \vec{b})$$

$$= -\omega^2 (\cos \omega t \vec{a} + \sin \omega t \vec{b}) = -\omega^2 \vec{r}$$

$$\Rightarrow \frac{d^2 \vec{r}}{dt^2} + \omega^2 \vec{r} = 0$$

(ii) $\vec{r} \times \frac{d\vec{r}}{dt} = (\cos \omega t \vec{a} + \sin \omega t \vec{b}) \times (-\omega \sin \omega t \vec{a} + \omega \cos \omega t \vec{b})$

$$= (\omega \cos^2 \omega t) \vec{a} \times \vec{b} + (\omega \sin^2 \omega t) \vec{a} \times \vec{b}$$

$$(\text{since } \vec{a} \times \vec{b} = -\vec{b} \times \vec{a})$$

$$= \omega (\cos^2 \omega t + \sin^2 \omega t) \vec{a} \times \vec{b} = \omega \vec{a} \times \vec{b}$$

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Ques 23 If $\vec{r} = (\cos nt)\hat{i} + (\sin nt)\hat{j}$ where n is a constant and t varies then show that.

Sol - $\vec{r} \times \frac{d\vec{r}}{dt} = n\hat{k}$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt} (\cos nt)\hat{i} + \frac{d}{dt} (\sin nt)\hat{j} = -n \sin nt \hat{i} + n \cos nt \hat{j}$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos nt & \sin nt & 0 \\ -n \sin nt & n \cos nt & 0 \end{vmatrix}$$

$$= (0-0)\hat{i} + (0-0)\hat{j} + (n \cos^2 nt + n \sin^2 nt)\hat{k}$$

$$= 0 + 0 + n(\cos^2 nt + \sin^2 nt)\hat{k} = n(1)\hat{k} = n\hat{k}$$

Ques 24. A particle moves along the curve $x=t^3+1$, $y=t^2$, $z=2t+5$, where t is time. Find the components of its velocity and acceleration at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$.

Sol - $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt} (x\hat{i} + y\hat{j} + z\hat{k}) = \frac{d}{dt} \{ (t^3+1)\hat{i} + t^2\hat{j} + (2t+5)\hat{k} \}$

$$= 3\hat{i} + 2t\hat{j} + 2\hat{k} \text{ at } t=1$$

Now unit vector in the direction of $\hat{i}+\hat{j}+3\hat{k}$ is

$$= \frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{11}}$$

the component of velocity at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$ is

$$(3\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}} = \frac{3+2+6}{\sqrt{11}} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

Acce. $a = \frac{d^2r}{dt^2} = 6\hat{i} + 2\hat{j} = 6\hat{i} + 2\hat{j}$ at $t=1$.

the component of acceleration at $t=1$ in the direction of $\hat{i} + \hat{j} + 3\hat{k}$.

$$(6\hat{i} + 2\hat{j}) \cdot \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}} = \frac{6+2}{\sqrt{11}} = \frac{8}{\sqrt{11}}$$

Ques 25 If $r = t^2\hat{i} + t\hat{j}$ and $s = 2t\hat{i} - t\hat{j} + t^2\hat{k}$. then evaluate

$$\int_0^1 (r \times s) dt$$

Sol
Here

$$r = t^2\hat{i} + t\hat{j} \quad \text{and} \quad s = 2t\hat{i} - t\hat{j} + t^2\hat{k} \quad \text{So.}$$

$$r \times s = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & t & 0 \\ 2t & -t & t^2 \end{vmatrix} = t^3\hat{i} - t^4\hat{j} - (t^3 + 2t^2)\hat{k}$$

$$\int_0^1 (r \times s) dt = \int_0^1 \{ t^3\hat{i} - t^4\hat{j} - (t^3 + 2t^2)\hat{k} \} dt$$

$$= \left[\frac{1}{4}\hat{i} - \frac{1}{5}\hat{j} - \left(\frac{1}{4} + \frac{2}{3} \right)\hat{k} \right] - 0 = \frac{1}{4}\hat{i} - \frac{1}{5}\hat{j} - \frac{11}{12}\hat{k}$$

$$= \frac{1}{4}\hat{i} - \frac{1}{5}\hat{j} - \frac{11}{12}\hat{k}$$

Ques 26 The nec. and suff. condition for a scalar point fun. to be constant is that $\nabla f = 0$.

Sol - for condition is neces.

Suppose $f(x, y, z) = \text{const.}$

$$\Rightarrow \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$$

$$\text{therefore } \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

Hence the condition is neces.

for condition is suff. Suppose $\nabla f = 0$

$$\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} = 0 \Rightarrow \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$$

$\Rightarrow f$ is constant $\Rightarrow$ condition is sufficient

Ques 27 Discuss the existence of $\lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$ doesn't exist

Sol - Expansion $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$

using this $\lim_{x \rightarrow 0} \frac{1 + \frac{1}{x} + \frac{1}{x^2} \frac{1}{2} + \dots - 1}{e^{1/x} + 1}$

$$\lim_{x \rightarrow 0} \frac{1}{x} \rightarrow \infty \quad e^{1/x} \rightarrow 0 \quad \text{L.H.L } \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \frac{0 - 1}{0 + 1} = -1$$

Ques 28 $\iint xy \, dx \, dy$ over the region in the 1st quadrant for which $x + y \leq 1$

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$$I = \int_0^1 \int_{y=0}^{1-x} xy \, dy \, dx = \int_0^1 x \left[\frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 x \frac{(1-x)^2}{2} dx = \frac{1}{2} \int_0^1 (x - 2x^2 + x^3) dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 = \frac{1}{2} \left[\frac{6-8+3}{12} \right] = \frac{1}{2} \cdot \frac{1}{12}$$

$$I = \frac{1}{24}$$

Ques (29) Find the constants a, b, c so that the vector $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

Sol.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$\vec{F}$ is irrotational
means $\nabla \times \vec{F} = 0$

$$= (c+1)\hat{i} - (4-a)\hat{j} + (b-2)\hat{k}$$

$$\Rightarrow (c+1)\hat{i} + (-4+a)\hat{j} + (b-2)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$c+1=0, \quad 4-a=0 \quad \text{and} \quad b-2=0$$

$$a=4, \quad b=2, \quad c=-1$$

Ques (30) State Green's Theorem.

Sol. If $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous fun. over a region R bounded by simple closed curve C in xy -plane, then.

$$\oint_C (\phi dx + \psi dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dxdy$$

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